21-301 Combinatorics
Homework 10
Due: Monday, November 15

1. Consider the following take-away game: $p \geq 3$ is a prime number. There is a pile of n chips. A move consists of removing p^k chips for some $k \geq 1$. Prove that the Sprague-Grundy numbers $g(n)$ for $n \geq 0$ are given by

$$g(n) = \begin{cases} 0 & n = 0, 1, \ldots, p-1 \mod 2p \\ 1 & n = p, p+1, \ldots, 2p-1 \mod 2p \end{cases}.$$

Solution: We verify this by induction. It is trivially true for $n \leq 2p$. For $n > 2p$ we have $g(n) = \text{mex}\{g(n-p), g(n-p^2), \ldots, \}$. Observe that if $k \geq 2$ then $p^k = p(p^{k-1}-1)+p$ and so $p^k \mod 2p = p$. It follows that $g(n) = \text{mex}\{g(n-p)\}$ and the induction step follows.

2. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. (For the purposes of this question, a proper factor of n, is any factor, including 1, that is strictly less than n). The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.

Solution: n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor of n is also odd.

3. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.