9\08\10

Recurrence Relations

We have some "unknown" sequence $a_0, a_1, \ldots$ $a_n = \#$ of certain objects of "size" n, ...

Goal is to find the value of a_n.

A recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \dots a_1, a_0)$$

a_0 is given

$a_1 = f_1(a_0)$

$a_2 = f_2(a_1, a_0)$

$\vdots$

Recurrence + initial values

fixes

$a_0, a_1, \dots a_n, \dots$

Linear Recurrence

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2$$

$$a_0 = a_1 = \underline{1}$$

$$1, 1, 2, 3, 5, 8, 13, \cdots$$

$$b_n = |B_n| = \{ x \in \{a,b,c\}^n : \text{aa does not occur in } x \}$$

$$B_1 = \{ a, b, c \}$$

$$B_2 = \{ ab, ac, ba, bb, bc, ca, cb, cc \}$$

$$\vdots$$

$$b_n = 2b_{n-1} + 2b_{n-2}$$

$$b_n = 2b_{n-1} + 2b_{n-2}$$

$$|B_n| = |B_n'| + |B_n''|$$

$x \in B_n$ such that $x_1 \neq a$

$x \in B_n$: $x_1 = a$

$B_n' = b \ldots\ldots$

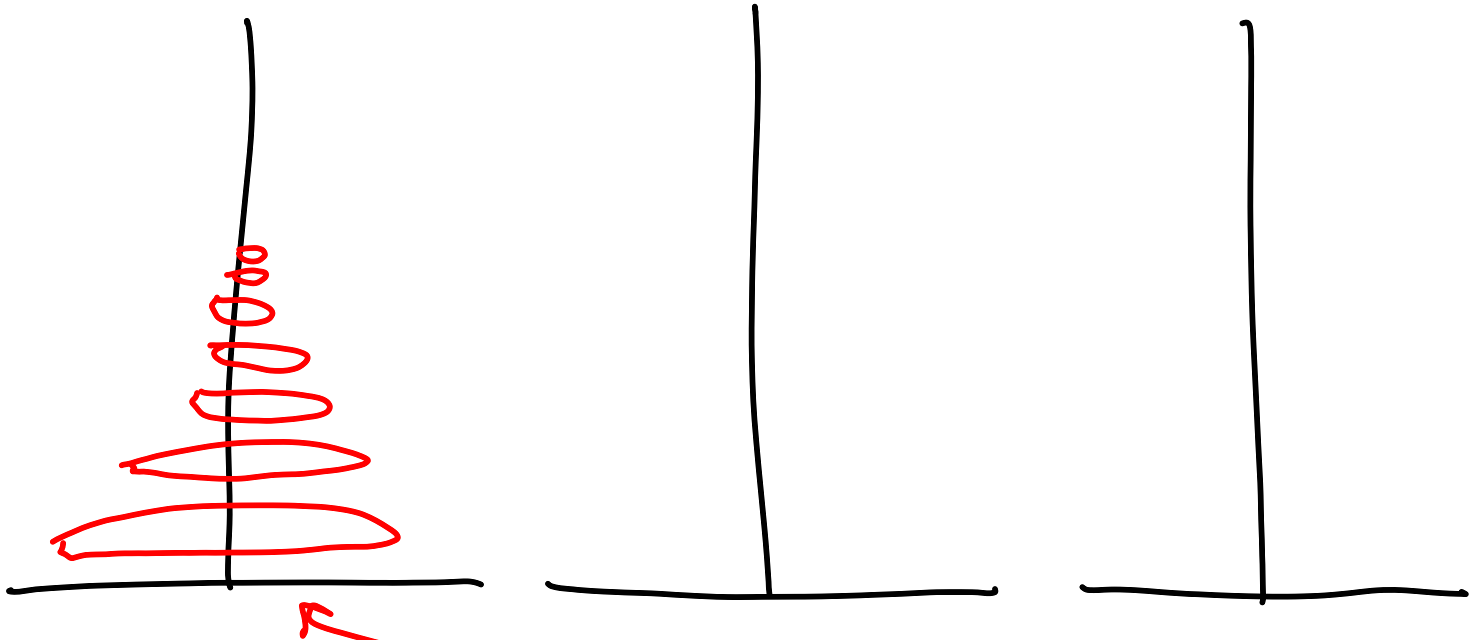
or $c \ldots\ldots$

any member of B_{n-1} can be used to complete

$ab \ldots\ldots$

$ac \ldots\ldots$

any B_{n-2}



Aim, move rings from to

Move one ring at a time.

<u>Never</u> place a larger ring on a smaller one.

H_n = minimum number moves needed.

$H_1 = 1$

H_n: (i) First move $n-1$ rings onto peg 2

(ii) Move big ring to peg 2

(iii) Move rings from peg 2 to peg 3.

$H_n = H_{n-1} + 1 + H_{n-1}$

A has n dollars.

Every day, A buys

Bun \$1

IceCream \$2

Pastry \$2

(His) her purchases define a string over $\{B, I, P\}$. U_n = # possibilities.

$$= U_{n,B} + U_{n,I} + U_{n,P}$$

ways if you buy Bun on first day

$$= U_{n-1} + U_{n-2} + U_{n-2}$$

Generating Functions

From $a_0 \quad a_1 \quad a_2 \quad a_3 \cdots \quad a_n \cdots$

we define

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

sequence $\Longrightarrow$ function $\Longrightarrow$ sequence

calculus

Taylor Series

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$1 \quad 1 \quad 1 \quad \dots \quad 1$$

$$a_n = 1$$

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n +$$

$$1 \quad 2 \quad 3 \quad \quad n+1$$

$$a_n = n+1$$

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + \dots$$

$$a_n = n$$

$$a_n - 10a_{n-1} + 25a_{n-2} = 0 \quad n \geq 2$$

$$a_0 = a_1 = 1$$

$$\sum_{n=2}^{\infty} (a_n - 10a_{n-1} + 25a_{n-2})x^n = 0$$

$$= \sum_{n=2}^{\infty} a_n x^n - 10\sum_{n=2}^{\infty} a_{n-1} x^n + 25\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$= \underbrace{\sum_{n=2}^{\infty} a_n x^n}_{} - 10 \underbrace{\sum_{n=2}^{\infty} a_{n-1} x^n}_{} + 25 \underbrace{\sum_{n=2}^{\infty} a_{n-2} x^n}_{} = 0$$

$$a_2 x^2 + a_3 x^3 + \dots$$

$$= a(x) - a_0 - a_1 x$$

$$= a(x) - 1 - x$$

$$a_1 x^2 + a_2 x^3 + \dots$$

$$= x(a_1 x + a_2 x^2 + \dots)$$

$$= x(a(x) - 1)$$

$$a_0 x^2 + a_1 x^3 + \dots$$

$$= x^2 a(x)$$

$$[a(x) - 1 - x] - 10x(a(x) - 1) + 25x^2 a(x) = 0$$

$$[a(x)-1-x]-10x(a(x)-1)+25x^2a(x)=0$$

$$a(x)[1-10x+25x^2]=1-9x$$

$$a(x)=\frac{1-9x}{1-10x+25x^2}$$

$$=\frac{1-9x}{(1-5x)^2}$$

$$=\frac{A}{1-5x}+\frac{B}{(1-5x)^2}$$

figure out

A, B

$$= \frac{1-9x}{(1-5x)^2}$$

$$= \frac{A}{1-5x} + \frac{B}{(1-5x)^2}$$

$$A(1-5x) + B = 1-9x \quad \forall x$$

$$A = 9/5 \qquad B = -\frac{4}{5}$$

$$a(x) = \frac{9}{5}\sum_{n=0}^{\infty}(5x)^n - \frac{4}{5}\sum_{n=0}^{\infty}(n+1)(5x)^n$$

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$$a_n = \frac{9}{5} * 5^n - \frac{4}{5} * (n+1) * 5^n$$