9\29\10

Claim $E(X+Y) = E(X) + E(Y)$

when X, Y are random variables

No independence necessary.

If X, Y are independent then

$$E(XY) = E(X)\,E(Y)$$

Binomial $B(n,p)$

$$E(B(n,p)) = ?$$

$$B(n,p) = X_1 + X_2 + \cdots + X_n$$

where

$$X_j = \begin{cases} 1 & j\text{'th coin is } H \\ 0 & \text{otherwise} \end{cases}$$

$$E(B(n,p)) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= (1 \times p + 0 \times (1-p)) + \cdots = np$$

Same probability space.

$E($# of occurrences of HTH in the sequence$)$

$$\# = X_1 + X_2 + \cdots + X_{n-2}$$

where

$$X_i = \begin{cases} 1 & \text{if } i^{th} = H,\ i+1^{th} = T,\ i+2 = T \\ 0 & \text{otherwise} \end{cases}$$

$$E(\#) = E(X_1) + E(X_2) + \cdots + E(X_{n-2})$$

$$= \left(1 \times p^2(1-p) + 0\right) + \cdots = (n-2)p^2(1-p)$$

m indistinguishable balls,
n colors

$Z_i = 1 \iff$ color i gets used

$Z = Z_1 + \cdots + Z_n$
$= \#$ colors used.

$E(Z_1) = \Pr(Z_1 \neq 0)$
$= 1 - \Pr(Z_1 = 0)$

$$\Pr(Z_1 = 0) = \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}$$

ways when 1 not used

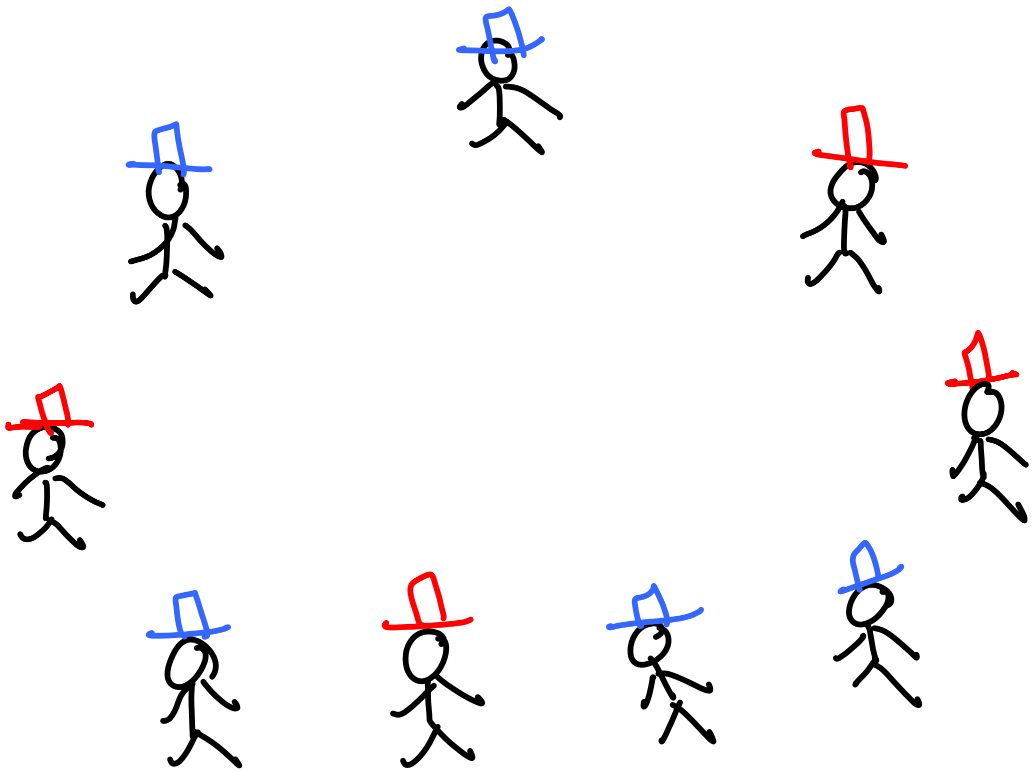
Total # of ways of coloring

$$= \frac{n-1}{n+m-1}$$

$$E(Z) = n\left(1 - \frac{n-1}{n+m-1}\right)$$

$$= \frac{mn}{n+m-1}$$

Hat Problem



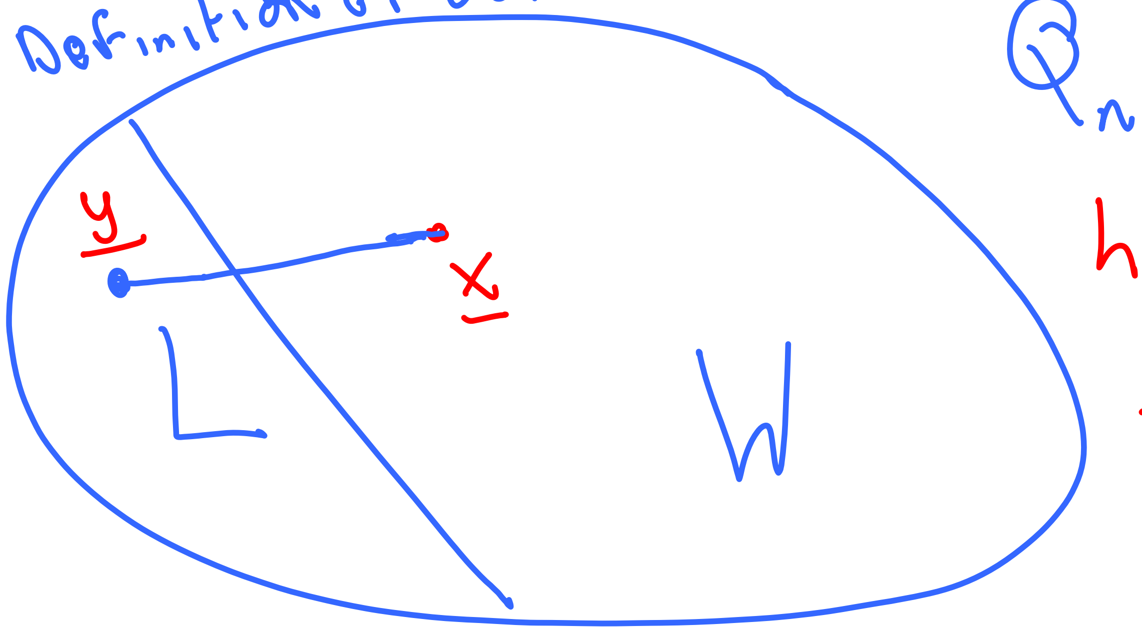
Color of each hat is RANDOM

$Q_n = \{0, 1\}^n$ — an assignment of hat colors is in Q_n

0 : Red, 1 : Blue.

$Q_n = W \cup L$

Definition of cover



Hamming Distance

$h(\underline{x}, \underline{y}) = 1$

$\# i : x_i \neq y_i$

There are many covers.

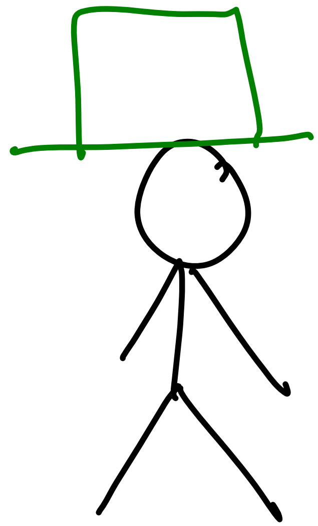
e.g. $L = \{ \underline{x} : x_1 = 0 \}$

Our people with the hats agree on a cover L.

Going to assume that $\underline{x} \in W$

Hat coloring

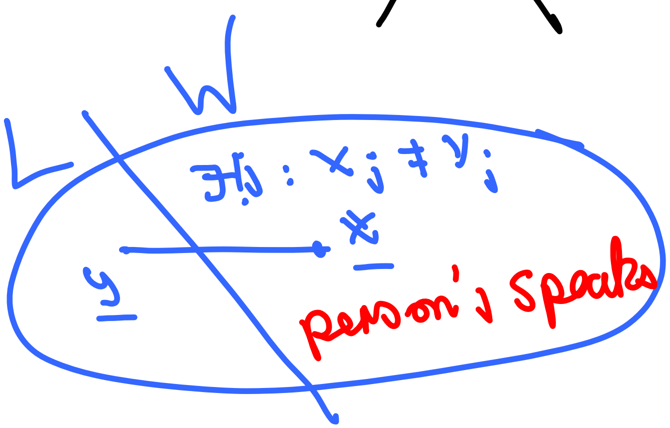
Who speaks?



$\underline{x} \in W$

If R or B keeps $\underline{x}$ in W
keep quiet

If exactly one puts you in W,
Speak up.



If $\underline{x} \in W$ there is at least one person

There exists L such that:

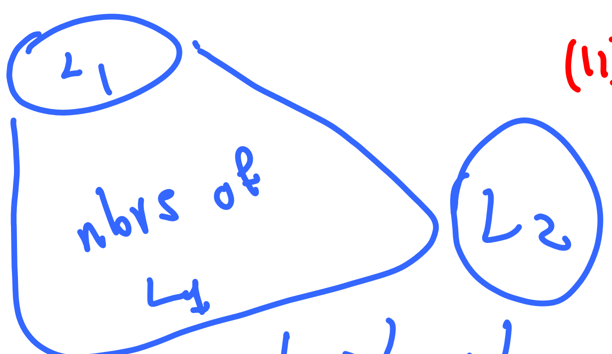
$$|L| \le 2^n \times \frac{2\log_e n}{n}$$

<u>Proof</u>

$$p = \frac{\ln n}{n}$$

(i) Put $y \in Q_n$ into L_1 with probability p

(ii) $L_2 = y$ which not Hamming distance ≤ 1 from L_1



$L = L_1 \cup L_2$

$$E(|L|) \leq E(|L_1|) + E(|L_2|)$$

$$2^n p \qquad\qquad 2^n (1-p)^{n+1}$$

Not in L_1 and no nbr in in L_1

$$\leq 2^n p + 2^n e^{-np}$$

$$\boxed{(1+x) \leq e^x, \ \forall x}$$

$$\leq 2^n \cdot \frac{2 \log n}{n}.$$