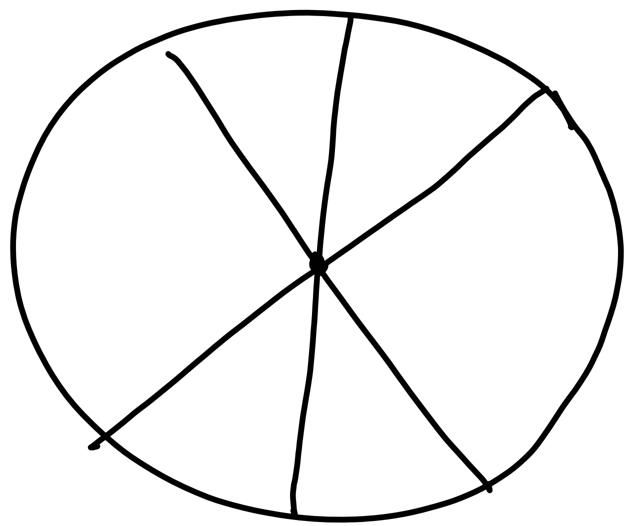
11\15\10



Rotatable

2 colors, Red & Blue.

How many distinct colorings?

R	B
B	B

is the same coloring as

B	B
B	R

X is a set (of colorings).

G is a set of permutations on X.

Consider a rotation 90° clockwise. π

$$\pi\left[\begin{array}{|c|c|}\hline R & B \\ \hline B & R \\ \hline\end{array}\right] = \begin{array}{|c|c|}\hline B & R \\ \hline R & B \\ \hline\end{array}$$

$|X| = 16$

Two colorings x_1, x_2 are "the same" if there exists $g \in G$ such that

$$g(x_1) = x_2.$$

G is a group

(I) G has an identity 1_X $(x) = x$ $\forall x \in X$

(II) G is associative

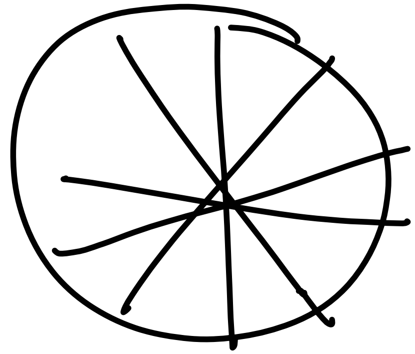
$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

composition

(III) $g^{-1} \in G$ i.e. $\forall g, \exists g^{-1}$

s.t. $g^{-1} \circ g = 1_X$

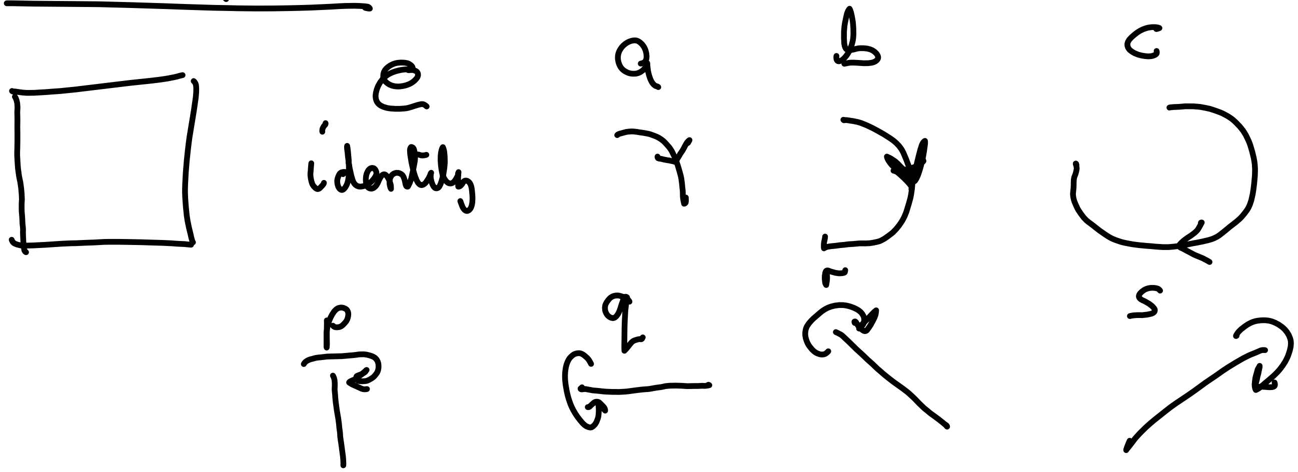
Example 1



$G = (e_0, e_1, \ldots, e_{n-1})$ where

e_i = rotation by $\frac{2i\pi}{n}$

Example 2



Orbits

$g(x)$

$x \in X$, $\mathcal{O}_x = \{ y \in X : \exists g \text{ s.t. } g * x = y \}$

orbit of x

Lemma

Orbits partition X

Proof $x = 1_X * x \Rightarrow x \in \mathcal{O}_x$.

Next, show that if $\mathcal{O}_x \cap \mathcal{O}_y \neq \emptyset$ then

$\mathcal{O}_x = \mathcal{O}_y$

Suppose $O_x \cap O_y \neq \emptyset$

$\Longrightarrow \exists\ g_1, g_2$ such that

$$g_1 * x = g_2 * y$$

$$\Downarrow$$

$$x = (g_1^{-1} \circ g_2) * y$$

So $\quad g * x = \left(g \circ (g_1^{-1} \circ g_2)\right) * y$

$O_x \subseteq O_y$

Problem: how many orbits are there?

$$S_x = \{ g : g * x = x \}$$

$$x = \begin{array}{|c|c|}\hline R & R \\ \hline R & R \\ \hline\end{array} \Rightarrow S_x = G$$

$$x = \begin{array}{|c|c|}\hline R & B \\ \hline B & R \\ \hline\end{array} \Rightarrow S_x = e, b, r, s$$

Claim

$x \in X$ then $|O_x||S_x| = |G|$ $(*)$

$$\Downarrow$$

$\nu_{X,G} = \#$ orbits, then

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

$$\nu_{X,G} = \sum_{x \in X} \frac{1}{|O_x|} \overset{(*)}{=} \sum_{x \in X} \frac{|S_x|}{|G|}$$

Proof of $|O_x||S_x| = |G|$

Fix $x \in X$

$$g_1 \sim g_2 \quad \text{iff} \quad g_1 * x = g_2 * x$$

Equivalence classes are $A_1, A_2, \ldots, A_m$

$|O_x|$

To show $|A_i| = |S_x|$

Fix i and $g \in A_i$ and now count all $h \in A_i$

$$h \in A_i \Leftrightarrow g * x = h * x \Leftrightarrow g^{-1}h \in S_x$$

$$\Longrightarrow \# h = |S_x|.$$