9\18\09

$a_0, a_1, \ldots, a_n, \ldots$

$a_0 = 1, \quad a_1 = 9$

$n \geq 2$: $a_n - 6a_{n-1} + 9a_{n-2} = 0$

Solution:

(i) Find $a(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$

(ii) Figure out a_n from this

$$\sum_{n=2}^{\infty}\left(a_n - 6a_{n-1} + 9a_{n-2}\right) \times x^n = 0$$

$$|||$$

$$\underbrace{\sum_{n=2}^{\infty} a_n x^n}_{a(x) - a_0 - a_1 x} - 6\underbrace{\sum_{n=2}^{\infty} a_{n-1} x^n}_{x\sum_{n=2}^{\infty} a_{n-1}x^{n-1}} + 9\underbrace{\sum_{n=2}^{\infty} a_{n-2} x^n}_{x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}} = 0$$

$$= a(x) - 1 - 9x$$

$$= x\sum_{m=1}^{\infty} a_m x^m$$

$$= x\left(a(x) - 1\right)$$

$$= x^2 a(x)$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) = 1 + 3x$$

or

$$a(x) = \frac{1+3x}{1-6x+9x^2}$$

$$a_n = [x^n]\,\frac{1+3x}{1-6x+9x^2}$$

$$\frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2} \quad (*)$$

Find A, B s.t. $\forall x$: $= \dfrac{A}{1-3x} + \dfrac{B}{(1-3x)^2} \quad (**)$

So multiply $(*)$ & $(**)$ by $(1-3x)^2$

$$1+3x = A(1-3x) + \underline{B} \quad \forall x$$

$$= \underbrace{(A+B)}_{1} \underbrace{- 3Ax}_{+3x}$$

$B = 2 \qquad A = -1$

$$a(x) = -\frac{1}{1-3x} + \frac{2}{(1-3x)^2}$$

$$= -\sum_{n=0}^{\infty} 3^n x^n + 2\sum_{n=0}^{\infty} (n+1)\, 3^n x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{3^n\left(-1 + 2(n+1)\right)}_{a_n} x^n$$

$$a_n - 3a_{n-1} + 2a_{n-2} = n+1$$

$n \geq 2$

$a_0 = 1, \quad a_1 = 4$

$$\sum_{n=2}^{\infty} (a_n - 3a_{n-1} + 2a_{n-2}) x^n = \sum_{n=2}^{\infty} (n+1) x^n$$

$$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} (n+1) x^n$$

$$a(x) - 1 - 4x - 3x(a(x) - 1) + 2x^2 a(x) = \frac{1}{(1-x)^2} - 1 - 2x$$

$$a(x)-1-4x-3x(a(x)-1)+2x^2a(x)=\frac{1}{(1-x)^2}-1-2x$$

$$a(x)\underbrace{(1-3x+2x^2)}_{(1-2x)(1-x)}=\frac{1}{(1-x)^2}-1-2x+1+4x-3x$$

$$=\frac{1}{(1-x)^2}-x$$

$$a(x)=\frac{1}{(1-x)^3(1-2x)}-\frac{x}{(1-2x)(1-x)}$$

$$=\frac{A}{(1-x)^3}+\frac{B}{(1-x)^2}+\frac{C}{1-x}+\frac{D}{1-2x}$$

We can figure out A, B, C, D later

$$\frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{1-2x}$$

$$= A \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$$

$$+ B \sum_{n=0}^{\infty} (n+1) x^n$$

$$+ C \sum_{n=0}^{\infty} x^n$$

$$+ D \sum_{n=0}^{\infty} 2^n x^n$$

$$a_n = \frac{A\,n(n+1)}{2} + B(n+1) + C + D \times 2^n$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \cdots$$

$$-\binom{-3}{1} \qquad \binom{-3}{2} \qquad \cdots$$

$$\binom{-3}{n} = \frac{(-3)(-4)\cdots(-3-n+1)}{n!} = (-1)^n \cdot \frac{(n+2)!}{2n!}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$a(x) = \frac{1}{(1-x)^3(1-2x)} - \frac{x}{(1-2x)(1-x)}$$

$$= \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{1-2x}$$

So, on multiplying through by $(1-x)^3(1-2x)$

$$1 - x(1-x)^2 = A(1-2x) + B(1-x)(1-2x) + C(1-x)^2(1-2x) + D(1-x)^3$$

Putting $x=1$ we get

$$1 = -A \quad \text{or} \quad A = -1$$

Putting $x = \frac{1}{2}$ we get

$$1 = D/4 \quad \text{or} \quad D = 4$$

$$1 - x(1-x)^2 = A(1-2x) + B(1-x)(1-2x) + C(1-x)^2(1-2x) + D(1-x)^3$$

Putting $x = 1$ we get

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$$1 = D/8 \quad \text{or} \quad D = 8$$

Putting $x = 0$ we get

$$1 = A + B + C + D = B + C + 7$$

Examing the coefficient of x^3 on both sides we get

$$-1 = -2C - D \quad \text{or} \quad C = -7/2$$

$$B = -9/2$$