10\2\09

Secretary Problem

n people want a job.

PAT must choose one

People appear in a random order.

Pat can evaluate a person correctly.

So if she could look at all of them at the same time, then she could choose best.

People appear one by one and she must her decision to reject or accept immediately.

She wants a strategy that gives her a good chance of picking the best

Strategy:

Choose $m \leq n$.

(I) Interview first m.

Reject them all.

(II) For $i = m+1, \dots, n$ accept person i if they are the best in $\{1, 2, \dots, i\}$

$S \equiv \{\text{Pat chooses best}\}$

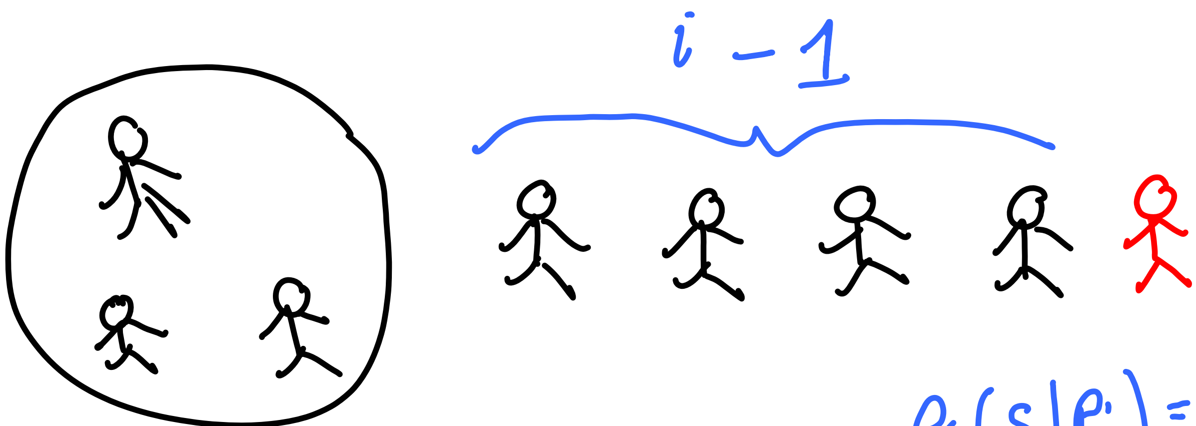
$P_i \equiv \{i\text{th person is the best}\}$

$$\Pr(S) = \sum_{i=1}^{n} \Pr(S \mid P_i)\underbrace{\Pr(P_i)}_{1/n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Pr(S \mid P_i)$$

$$\Pr(S \mid P_i) = 0 \quad \text{if } i \leq m.$$

Suppose $i > m$



m people

$$\Pr(S \mid P_i) = \frac{m}{i-1} \qquad i > m$$

$S \mid P_i \iff$ best of first $i-1$ in first m

$$P(S) = \frac{1}{n}\sum_{i=m+1}^{n}\frac{m}{i-1}$$

$$= \frac{m}{n}\sum_{i=m+1}^{n}\frac{1}{i-1}$$

Assume $m \approx \alpha n$ for some $\alpha < 1$

$$\approx \frac{m}{n}\left[\log n - \log m\right]$$

$$= \alpha \log 1/\alpha$$

$$P(S) = \underbrace{\alpha \log 1/\alpha}_{f(\alpha)} \quad *$$

$$f'(\alpha) = -\log \alpha - 1$$

$$f''(\alpha) = -\frac{1}{\alpha} < 0$$

$f'(\alpha) = 0$ when $\alpha = 1/e$

$* \quad -\alpha \log \alpha$

When $\alpha = 1/e$

$$Pr(S) = 1/e.$$

Random Variables

A function $\zeta : \Omega \to \mathbb{R}$ is called a random variable.

Two Dice (x_1, x_2)

$\zeta = x_1 + x_2$

$$p_k = P_r(\zeta = k)$$

<u>Dice</u>

k	2	3	4	5	6	7 ...	12
p_k	$\frac{1}{36}$					$\frac{1}{6}$	

$\Omega = \{ m$ indistinguishable balls, n colors $\}$

$m=3,\ n=2$

$\zeta=2$: RRB, RBB, ...

$\zeta=1$: RRR, BBB

ζ = # of colors actually used

choose colors

colorings with k colors, each used at least once.

$$P_k = \frac{\binom{n}{k}\binom{m-1}{k-1}}{\binom{n+m-1}{n-1}}$$

Binomial Random Variable

n coin tosses.

$p = P[\text{Heads}]$

$\omega = T\,H\,H\,T\,T\,H\,H\,T\,T\,H$

$B_{n,p}(\omega) = \#$ heads

$$p_k = P_r(B_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

coins which come up heads

Expectation (Average)

Z is a random variable.

$$E(Z) = \sum_{\omega \in \Omega} Z(\omega) P(\omega)$$ "weighted average"

e.g.

$P(\omega) = \frac{1}{|\Omega|}$: $E(Z) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} Z(\omega)$

Uniform