21-301 Combinatorics
Homework 9
Due: Wednesday, November 19

1. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. (For the purposes of this question, a proper factor of n, is any factor, including 1, that is strictly less than n). The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.

   Solution: n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor of n is also odd.

2. Consider the following game: There is a single pile of n chips. A move consists of removing (i) any *even* number of chips provided it is not the whole pile, or (ii) the whole pile, but only if it has 2 (mod 3) chips. The terminal positions are zero and one. Determine the Sprague-Grundy numbers of each pile size.
   (Compute the first 15 numbers and see if you can see see a pattern.)

   Solution: The Sprague-Grundy function g is given by

   $$g(0) = g(1) = g(4) = 0 \; and \; g(2) = g(3) = 1 \; and \; g(k) = \lceil n/2 \rceil - 1 \; for \; n \geq 5.$$

   We verify the last claim by induction. It can be checked for $n = 5, 6$. Suppose next that $k > 3$. Then if $*$ is $g(0) = 0$ for $n \mod 3 = 2$ and not there otherwise,

$$\begin{aligned}
g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(6), g(4), g(2), *\} \\
&= mex\{k-2, k-3, \ldots, 2, 0, 1, *\} \\
&= k-1. \\
g(2k+1) &= mex\{g(2k-1), g(2k-3), \ldots, g(5), g(3), g(1), *\} \\
&= mex\{k-1, k-2, \ldots, 2, 1, 0, *\} \\
&= k.
\end{aligned}$$

3. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.