21-301 Combinatorics
Homework 8
Due: Wednesday, November 12

1. Suppose we 2-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.

   Solution: Assume w.l.o.g. that triangle $(1,2,3)$ is Red and that $(4,5,6)$ is not Red and in particular that edge $(4,5)$ is Blue. If $x=4,5$ or 6 then there can be at most one Red edge joining x to $1,2,3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4,5$ to $1,2,3$. So there must be $x \in \{1,2,3\}$ such that both $(x,4)$ and $(x,5)$ are Blue. But then triangle $(x,4,5)$ is Blue.

2. Let $r_n = r(3,3,\ldots,3)$ be the minimum integer such that if we n-color the edges of the complete graph K_N there is a monochromatic triangle.

   (a) Show that $r_n \leq n(r_{n-1}-1)+2$.

   Solution: Let $N = n(r_{n-1}-1)+2$ and consider an n-colouring σ of the edges of K_N. Now consider the $N-1$ edges incident to vertex N. There must be a color, n say, that is used at least r_{n-1} times, Pigeon Hole Principle. Now let $V \subseteq [N-1]$ denote the set of vertices v for which the edge $\{v,N\}$ is colored n. Consider the coloring of the edges of V induced by σ. If one of these $\{v_1,v_2\}$ has color N then it makes a triangle v_1,v_2,N with 3 edges colored n. Otherwise the edges of V only use $n-1$ colors and since $|V| \geq r_{n-1}$ we see by induction that V contains a mono-chromatic triangle.

   (b) Using $r_2 = 6$, show that $r_n \leq \lfloor n!e \rfloor + 1$.

   Solution: Divide the inequality by $n!$ and putting $s_n = r_n/n!$ we obtain

$$s_n \leq s_{n-1} - \frac{1}{(n-1)!} + \frac{2}{n!}. \tag{1}$$

We write this as

$$\begin{aligned} s_n - s_{n-1} &\leq -\frac{1}{(n-1)!} + \frac{2}{n!} \\ s_{n-1} - s_{n-2} &\leq -\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \\ &\vdots \\ s_3 - s_2 &\leq -\frac{1}{1!} + \frac{2}{2!} \end{aligned}$$

Summing gives

$$s_n - s_2 \leq -1 + \frac{1}{n!} + \sum_{k=2}^{n} \frac{1}{k!} \leq -1 + \frac{1}{n!} + e - 2.$$

Now $s_2 = 3$ and multiplying the above by $n!$ gives $r_n \leq n!e + 1$. We round down, as r_n is an integer.

3. Let G_1, G_2 be fixed graphs. Let $r(G_1, G_2)$ be the smallest integer such that if we two-color the edges of the complete graph K_N there is a Red copy of G_1 or a Blue copy of G_2, or both. Show that if P_3 is a path of length 3 and C_4 is a 4-cycle, then

$$r(P_3, P_3) = 5,\ r(P_3, C_4) = 5,\ r(C_4, C_4) = 6.$$

Solution:

$r(P_3, P_3)$**:**

K_4 is the union of an edge disjoint triangle and a copy of $K_{1,3}$ and so $r(P_3, P_3) > 4$.

Now consider a two coloring of the edges of K_5. There is a color used at least twice at vertex Red. So assume that edges (1,2),(1,3) are Red. If (2,4) is Red then (4,2,1,3) is Red and if (2,5) is Red then (5,2,1,3) is Red. So we can assume that (2,4),(2,5) are Blue. But then if (3,4) is Red we have (4,3,1,2) is Red and if (3,4) is Blue then (3,4,2,5) is Blue.

$r(P_3, C_4)$**:**

K_4 is the union of an edge disjoint triangle and a copy of $K_{1,3}$ and so $r(P_3, C_4) > 4$.

Now consider a two coloring of the edge of K_5. We can assume from $r(P_3, P_3) = 5$ that there is a Blue P_3, say (1,2,3,4) and that (1,4) is Red.

Consider the edges (2,5),(4,5). If they are both Red then (1,4,5,2) is Red. If they are both Blue then (2,3,4,5,2) is Blue. A similar argument deals with the case where (1,5) and (3,5) have the same color.

Assume next that (4,5) is Blue and (2,5) is Red. If (1,5) is Red then (2,5,1,4) is Red. So assume that (1,5) is Blue and (3,5) is Red. If now (1,3) is Red then so is (4,1,3,5) and if (1,3) is Blue then so is (1,3,4,5,1).

Finally, suppose that (4,5) is Red and (2,5) is Blue. If (3,5) is Red then (3,5,4,1) is Red. So assume that (3,5) is Blue and (1,5) is Red. If (1,3) is Red then so is (3,1,5,4) and if (1,3) is Blue then so is (1,3,5,2,1).

$r(C_4, C_4)$**:**

K_5 is the union of 2 edge disjoint C_5's and so $r(C_4, C_4) > 5$.

Now consider a two coloring of the edge of K_6. Consider the edges incident with 1. At least 3 must be the same color. Assume therefore that the edges (1,2),(1,3),(1,4) are all Red. It follows that neither of vertices 5 and 6 can have 2 Red edges joining them to 1,2,3. Suppose that edges (2,5),(3,5) are Blue. Then at most one of (6,2),(6,3) can be Blue. Otherwise we have the Blue (2,5,3,6,2). So assume that (3,6),(4,6) are Blue and (2,6),(4,5) are Red.

Suppose now that (5,6) is Blue. If (2,3) is Blue then so is (5,6,3,2,5). If (3,4) is Blue then so is (5,6,4,3,5). But now if (2,3) and (3,4) are Red then (1,2,3,4,1) is Red.

So we can assume that (5,6) is Red. If (2,4) is Red then (2,4,6,5,2) is Red. Suppose then that (2,4) Blue. If for example (2,3) is Blue then so is (2,3,6,4,2). So assume now that (2,3) and (3,4) both Red and then (1,2,3,4,1) is Red.

Somewhat longer than I antcipated!