21-301 Combinatorics
Homework 7
Due: Monday, November 3

1. Let $\chi(G)$ be the chromatic number of graph $G = (V, E)$. Let $\alpha(G), \kappa(G)$ denote the size of the largest independent set of G, clique of G respectively.

   Show that
   $$\chi(G) \geq \max\left\{\frac{|V|}{\alpha(G)}, \kappa(G)\right\}.$$
   Show further that $\chi(G)\chi(\bar{G}) \geq |V|$. Here $\bar{G}$ is the complement of G.

   Solution: If S is a clique of size s then we need at least s colors, just to color S. Thus $\chi(G) \geq \kappa(G)$. Observe next that in a proper coloring, a set of vertices of the same color must form an independent set. Thus a proper coloring partitions the vertex set into a set of color classes $C_1, C_2, \ldots, C_k$ where $|C_i| \leq \alpha(G)$ for $1 \leq i \leq k$. This implies that $k \geq |V|/\alpha(G)$.

   For the second part, we have
   $$\chi(G) \geq \frac{|V|}{\alpha(G)} = \frac{|V|}{\kappa(\bar{G})}.$$
   Hence,
   $$\chi(G)\chi(\bar{G}) \geq \chi(G)\kappa(\bar{G}) \geq \frac{|V|}{\kappa(\bar{G})}\kappa(\bar{G}) = |V|.$$

2. Given any sequence of n integers, positive or negative, not necessarily all different, show that some consecutive subsequence has the property that the sum of the members of the subsequence is a multiple of n.

   Solution: Let the sequence be $x_1, x_2, \ldots, x_n$ and let $s_i = x_1 + \cdots + x_i \mod n$ for $i = 1, 2, \ldots, n$. If there exists i with $s_i = 0$ then n divides $x_1 + \cdots + x_i$. Otherwise, $s_1, s_2, \ldots, s_n$ all take values in $[n-1]$. By the pigeon-hole principle, there exist $i < j$ such that $s_i = s_j$ and then n divides $x_{i+1} + \cdots + x_j$.

3. Suppose that $a_1, a_2, \ldots, a_n \in [n]$ and $b_1, b_2, \ldots, b_n \in [n]$. An interval I is a set of the form $\{i, i+1, \ldots, j\}$. Let $a_I = a_i + a_{i+1} + \cdots + a_j$ and

$b_I = b_i + b_{i+1} + \cdots + b_j$. Show that there exist intervals I, J such that $a_I = b_J$.

Solution: Without loss of generality, let us assume that

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i. \tag{1}$$

Now, for any j it is always possible to express $\sum_{i=1}^{j} b_i$ as $\sum_{i=1}^{j} b_i = \sum_{i=1}^{k} a_i + R_j$ $R_j \in [0, n-1]$ for some $k = k(j) \in [0, n]$. (When $k = 0$ this means that $\sum_{i=1}^{j} b_i \in [0, n-1]$). Indeed, either $\sum_{i=1}^{j} b_i < a_1$ and then we can directly see that $\sum_{i=1}^{j} b_i = R_j < a_1 \in [1, n]$. Otherwise, take the largest k such that $S_k = \sum_{i=1}^{j} b_j - \sum_{i=1}^{k} a_i \geq 0$. Now $k < n$ by assumption (1). If $S_k \geq n$ then $S_{k+1} \geq 0$, contradiction.

Consider the set of values R_j, for $j \in [1, n]$. If $R_j = 0$, for some j then we are done since $\sum_{i=1}^{j} b_i = \sum_{i=1}^{k(j)} a_i$. If not, there are only $n-1$ possible values for the n quantities $R_1, \ldots, R_n$ and so there exist $j_1 < j_2$ such that $R_{j_1} = R_{j_2}$. But then

$$\sum_{i=1}^{j_1} b_i - \sum_{i=1}^{k(j_1)} a_i = R_{j_1} = R_{j_2} = \sum_{i=1}^{j_2} b_i - \sum_{i=1}^{k(j_2)} a_i$$

and therefore,

$$\sum_{i=j_1}^{j_2} b_i = \sum_{i=k(j_1)}^{k(j_2)} a_i.$$