21-301 Combinatorics
Homework 6
Due: Monday, October 27

1. Using Cayley's formula, show that the graph obtained from K_n by deleting one edge has exactly $(n-2)n^{n-3}$ spanning trees.

   Solution: Fix an edge e say. Let n_e denote the number of trees containing e. Note first that n_e is independent of e. Denote this common value by ν. To answer the question, we must show that $\nu = 2n^{n-3}$. For a tree T and an edge e let

$$A(T,e) = \begin{cases} 1 & e \in T \\ 0 & e \notin T \end{cases}$$

   Then,

$$\sum_T \sum_e A(T,e) = \sum_T (n-1) = (n-1)n^{n-2}.$$

   Also,

$$\sum_e \sum_T A(T,e) = \sum_e n_e = \binom{n}{2}\nu.$$

   So,

$$\nu = \frac{(n-1)n^{n-2}}{\binom{n}{2}} = 2n^{n-3}$$

   as required.

2. Let $G = (V,E)$ be an r-regular graph with n vertices i.e. every vertex has degree r. $S \subseteq V$ is a *dominating set* if $w \notin S$ implies that there exists $v \in S$ for which $\{v,w\} \in E$. Show, by the probabilistic method, that G has a dominating set of size at most $\frac{1+\ln r}{r}n$.

   Solution: Let $p = \frac{\ln r}{r}$ and let S_1 be a random sub-set of V where each $v \in V$ is placed into S_1, independently with probability p. Let S_2 be the set of vertices that are not adjacent to any vertex of S_1. The set $S = S_1 \cup S_2$ is a dominating set.

$$\mathbf{E}(|S|) = \mathbf{E}(|S_1|) + \mathbf{E}(|S_2|) = np + n(1-p)^{r+1} \le np + ne^{-rp} \le \frac{1+\ln r}{r}n.$$

So there must be a dominating set of the required size.

3. Let $G = (V, E)$ be a graph with minimum degree at least three. Show that it contains a cycle of even length. (Hint: Consider a longest path).

   Solution: Let $P = (x = x_0, x_1, \ldots, x_k)$ be a longest path in G. Let $x_1, x_i, x_j, 1 < i < j$ be three neighbors of x. If i is odd then the cycle $(x_0, x_1, \ldots, x_i, x_0)$ has $i+1$ edges and is even and so we can assume that i, j are both even. But then the cycle $(x_0, x_i, x_{i+1}, \ldots, x_j, x_0)$ has $j-i+2$ edges and is even.