21-301 Combinatorics
Homework 5
Due: Monday, October 13

1. A box has m drawers; Drawer i contains g_i gold coins, s_i silver coins and ℓ_i lead coins, for $i = 1, 2, \ldots, m$. Assume that one drawer is selected randomly and that two randomly selected coins from that drawer turn out to have the same color. What is the probability that the chosen drawer is drawer 1?

   Solution: Let B_g, B_s, B_l be the events that both of the coins are gold, silver and lead respectively and let D_i be the event that drawer i is chosen. Let $B = B_g \cup B_s \cup B_l$. What we are asked for is

$$\Pr(D_1 \mid B) = \frac{\Pr(D_1 \cap B)}{\Pr(B)} = \frac{\Pr(D_1 \cap B_g) + \Pr(D_1 \cap B_s) + \Pr(D_1 \cap B_l)}{\Pr(B)}.$$

   Now

$$\begin{aligned}
\Pr(D_i \cap B_g) &= \frac{1}{m} \cdot \frac{g_i(g_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)} \\
\Pr(D_i \cap B_s) &= \frac{1}{m} \cdot \frac{s_i(s_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)} \\
\Pr(D_i \cap B_l) &= \frac{1}{m} \cdot \frac{l_i(l_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}
\end{aligned}$$

   and so

$$\Pr(B) = \sum_{i=1}^{m} \Pr(D_i \cap B) = \frac{1}{m} \sum_{i=1}^{m} \frac{g_i(g_i - 1) + s_i(s_i - 1) + l_i(l_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}.$$

   Therefore,

$$\Pr(D_1 \mid B) =$$
$$\frac{g_1(g_1 - 1) + s_1(s_1 - 1) + l_1(l_1 - 1)}{(g_1 + s_1 + \ell_1)(g_1 + s_1 + \ell_1 - 1)} \left(\sum_{i=1}^{m} \frac{g_i(g_i - 1) + s_i(s_i - 1) + l_i(l_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)} \right)^{-1}.$$

2. Let $m = \lfloor (8/7)^{n/3} \rfloor$. Show that there exist distinct sets $A_1, A_2, \ldots, A_m \subseteq [n]$ such that for all distinct $i, j, k \in [m]$ we have $A_i \cap A_j \not\subseteq A_k$.

   Solution: Let $A_1, \ldots, A_m$ be chosen randomly from $[n]$. Then for fixed i, j, k we have

$$\Pr(A_i \cap A_j \subseteq A_k) = \left(\frac{7}{8}\right)^n.$$

   Also,

$$\Pr(A_i = A_j) = \frac{1}{2^n}.$$

Let $\mathcal{D}$ be the event that the sets are not distinct and let $\mathcal{I}$ be the event that there exists i, j, k such that $A_i \cap A_j \subseteq A_k$. Then,

$$\Pr(\mathcal{D} \cup \mathcal{I}) \leq \binom{m}{2}\frac{1}{2^n} + 3\binom{m}{3}\left(\frac{7}{8}\right)^n < 1.$$

3. A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L-1]$ it makes a move to $i-1$ with probability $1/4$ and a move to $i+1$ with probability $3/4$. When at L it stops.

   Let E_k denote the expected number of visits to 0 if we started the walk at k.

   (a) Explain why

$$\begin{aligned} E_L &= 0 \\ E_0 &= 1 + E_1 \\ E_k &= \frac{1}{4}E_{k-1} + \frac{3}{4}E_{k+1} \qquad for\ 0 < k < L. \end{aligned}$$

   (b) Given that $E_k = \frac{A}{3^k} + B$ is a solution to your equations for some A, B, determine A, B and hence find E_0.

   Solution:

   $E_L = 0$ because the particle will not move from L. $E_0 = 1 + E_1$ because the particle must move to position 1 and then the expected number of extra visits to 0 will now be E_1. Finally,

$$E_k =$$
$$\mathbf{E}(\#\ visits | moves\ right)\Pr(moves\ right) + \mathbf{E}(\#\ visits | moves\ left)\Pr(moves\ left)$$
$$= \frac{1}{4}E_{k-1} + \frac{3}{4}E_{k+1}.$$

   $E_L = 0$ implies then that $\frac{A}{3^L} + B = 0$ and so $B = -\frac{A}{3^L}$.
   $E_0 = 1 + E_1$ implies then that $A + B = 1 + \frac{1}{3}A + B$ which implies that $A = 3/2$.
   Thus

$$E_0 = \frac{3 - 3^{1-L}}{2}.$$