21-301 Combinatorics
Homework 4
Due: Monday, October 6

1. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 3a_{n-1} + 2a_{n-2} = 4 \quad \text{for } n \geq 2$$

with initial conditions $a_0 = 2$ and $a_1 = 6$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty}(a_n + 3a_{n-1} + 2a_{n-2})x^n &= 4\sum_{n=2}^{\infty} x^n \\
a(x) - 2 - 6x + 3x(a(x) - 2) + 2x^2 a(x) &= \frac{4x^2}{1-x} \\
a(x)(1 + 3x + 2x^2) &= 2 + 12x + \frac{4x^2}{1-x}
\end{aligned}$$

$$\begin{aligned}
a(x) &= \frac{2+12x}{(1+x)(1+2x)} + \frac{4x^2}{(1-x)(1+x)(1+2x)} \\
&= \frac{2/3}{1-x} + \frac{8}{1+x} - \frac{20/3}{1+2x} \\
&= \sum_{n=0}^{\infty}\left(\frac{2}{3} + 8(-1)^n - \frac{20}{3}(-2)^n\right)x^n.
\end{aligned}$$

So

$$a_n = \frac{2}{3} + 8(-1)^n - \frac{20}{3}(-2)^n \qquad \text{for } n \geq 0.$$

2. Suppose that you are asked to multiply a collection of $m \times m$ matrices to form the product $A_1 A_2 \cdots A_{n+1}$. Let $C_0 = 1$ and let C_n be the number of ways to do this. For example if $n = 2$ then $d_n = 2$. We can compute $(A_1 A_2)A_3$ or $A_1(A_2 A_3)$. Show that

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Determine C_n.

Solution: The term $C_k C_{n-k}$ counts the number of ways of first computing the products $C_1 C_2 \cdots C_k$ and $C_{k+1} C_{k+2} \cdots C_{n+1}$ and then multiplying the two resulting matrices together. By summing over k we count all possible ways of doing the multiplication.

Let a_n be the number of solutions to the polygon triangulation problem. Thus $a_0 = 0, a_1 = a_2 = 1$ and $a_n = \sum_{k=0}^{n} a_k a_{n-k}$ for $n \geq 2$. We claim that $C_n = a_{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

We prove this by induction on n. It is clearly true for $n = 0$. So,

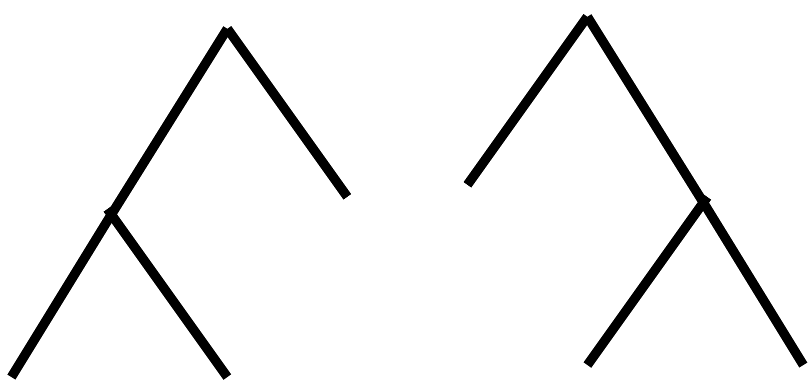
$$\begin{aligned}
C_{n+1} &= \sum_{k=0}^{n} C_k C_{n-k} \\
&= \sum_{k=0}^{n} a_{k+1} a_{n+1-k} \qquad \textit{by induction} \\
&= a_1 a_{n+1} + a_2 a_n + \cdots + a_{n+1} a_1 \qquad \textit{expanding sum} \\
&= a_0 a_{n+2} + a_1 a_{n+1} + a_2 a_n + \cdots + a_{n+1} a_1 + a_{n+2} a_0 \qquad \textit{since } a_0 = 0 \\
&= a_{n+2}.
\end{aligned}$$

This completes the inductive step.

3. Let T_n denote the number of binary trees with $n+1$ leaves. Show that

$$T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k}.$$

Determine T_n.



Solution: The product $T_k T_{n-k}$ is the number of trees whose left sub-tree has k vertices and whose right sub-tree has $n-k$ vertices. Together with the root, this makes $n+1$ vertices altogether.By summing over $k = 0$ to n we count allpossible trees. Now $T_0 = 1$ and T_n satisfies the same recurrence as C_n of Question 2. Thus $T_n = C_n = \frac{1}{n+1}\binom{2n}{n}$.