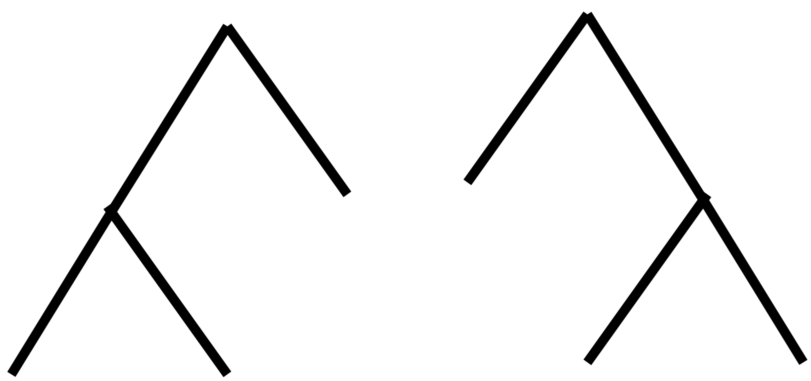
21-301 Combinatorics
Homework 4
Due: Monday, October 6

1. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation
$$a_n + 3a_{n-1} + 2a_{n-2} = 4 \quad \text{for } n \geq 2$$
with initial conditions $a_0 = 2$ and $a_1 = 6$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

2. Suppose that you are asked to multiply a collection of $m \times m$ matrices to form the product $A_1 A_2 \cdots A_{n+1}$. Let $C_0 = 1$ and let C_n be the number of ways to do this. For example $C_2 = 2$. We can compute $(A_1A_2)A_3$ or $A_1(A_2A_3)$. Show that
$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$
Determine C_n.

3. Let T_n denote the number of binary trees with $n+1$ leaves. Show that
$$T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k}.$$
Determine T_n.



$T_2 = 2$.