21-301 Combinatorics
Homework 3
Due: Monday, September 29

1. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form xx where $x = a, b$ satisfies the recurrence $b_0 = 1, b_1 = 3$ and
$$b_n = b_{n-1} + 2(b_{n-2} + \cdots + b_0) + 2b_0. \tag{1}$$
[Hint: Consider the number of sequences where the first c from the left is at position k.]

   Deduce from this that
$$b_n = 2b_{n-1} + b_{n-2}. \tag{2}$$
Solution: Let B_n denote the set of allowed sequences of length n. Suppose that c appears first in position k. Then the sequence starts $abab\cdots$ or $baba\cdots$, then c and then any sequence from B_{n-k}. Thus the number of such is $2b_{n-k}$ for $1 \leq k \leq n$. The extra $2b_0 = 2$ counts sequences without c.

   To get (2), subtract equation (1) with n replaced by $n-1$ from equation (1).

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and
$$\begin{aligned} b_n &= 2b_{n-1} + c_n & (3) \\ c_n &= c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} & (4) \end{aligned}$$
where c_n is the number of such sequences that start with a.

   Now find a recurrence only involving b_n, by using (3) to eliminate c_n from (4).

   Solution: There are $2b_{n-1}$ sequences of the required form that start with b or c. There are c_n sequences that start with a. This explains (3).

   There are c_{n-1} sequences that start with aa, b_{n-2} sequences that start with ac, c_{n-2} sequences that start with aba and b_{n-3} sequences that start with abb. This covers the possibilities for sequences starting with a.

   We have
$$b_n - 2b_{n-1} = b_{n-1} - 2b_{n-2} + b_{n-2} + b_{n-2} - 2b_{n-3} + b_{n-3}$$
and so
$$b_n = 3b_{n-1} - b_{n-3}.$$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation
$$a_n + 3a_{n-1} + 2a_{n-2} = n \quad \text{for } n \geq 2$$
with initial conditions $a_0 = 1$ and $a_1 = 3$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty}(a_n + 3a_{n-1} + 2a_{n-2})x^n &= \sum_{n=2}^{\infty} nx^n \\
a(x) - 1 - 3x + 3x(a(x) - 1) + 2x^2 a(x) &= \frac{x}{(1-x)^2} - x \\
a(x)(1 + 3x + 2x^2) &= \frac{x}{(1-x)^2} + 1 + 5x
\end{aligned}$$

$$\begin{aligned}
a(x) &= \frac{x}{(1+x)(1+2x)(1-x)^2} + \frac{1+5x}{(1+x)(1+2x)} \\
&= \frac{17/4}{1+x} + \frac{-31/9}{1+2x} + \frac{1/36}{1-x} + \frac{1/6}{(1-x)^2} \\
&= \sum_{n=0}^{\infty}\left(\frac{17}{4}(-1)^n - \frac{31}{9}(-2)^n + \frac{1}{36} + \frac{1}{6}(n+1)\right)x^n.
\end{aligned}$$

So

$$a_n = \frac{17}{4}(-1)^n - \frac{31}{9}(-2)^n + \frac{1}{36} + \frac{1}{6}(n+1) \qquad \text{for } n \geq 0.$$