21-301 Combinatorics
Homework 2
Due: Friday, September 12

1. Use induction to show that
$$\binom{n-1}{k} = \binom{n}{k} - \binom{n}{k-1} + \cdots \pm \binom{n}{0}.$$

Solution We use induction on k for a fixed n.
Base Case: $k = 0$. This is trivial, $\binom{k}{0} = \binom{k-1}{0}$.
Inductive Step: Suppose that the identity is true for some $k \geq 0$. Then
$$\begin{aligned}\binom{n}{k+1} - \binom{n}{k} + \cdots \pm \binom{n}{0} &= \binom{n}{k+1} - \left(\binom{n}{k} - \cdots \pm \binom{n}{0}\right) \\ &= \binom{n}{k+1} - \binom{n-1}{k} \qquad \textit{Induction} \\ &= \binom{n-1}{k+1}. \qquad \textit{Pascal's Triangle}\end{aligned}$$

2. **(a)** Let $\mathcal{S}_k$ denote the collection of k-sets $\{1 \leq i_1 < i_2 < \cdots < i_k \leq m-2\} \subseteq [m]$ such that $i_{t+1} - i_t \geq 2$ for $1 \leq t < k$. Show that
$$|\mathcal{S}_k| = \binom{m-2k}{k}.$$

(b) How many of the 3^n sequences $x_1 x_2 \cdots x_{2n}$, $x_i \in \{a, b, c\}$, $i = 1, 2, \ldots, n$ are there such that abc does not appear as a subsequence e.g. if $n = 6$ then we include $aabbcc$ in the count, but we exclude $aabcba$.

Solution (a) For a first argument, let $z_1 = i_1, z_2 = i_2 - i_1, \ldots, z_k = i_k - i_{k+1}, z_{k+1} = m - i_k$. We can count the number of choices for $z_1, z_2, \ldots, z_{k+1}$. But these are the soluitons to
$$z_1 + z_2 + \cdots + z_{k+1} = m,\ z_1 \geq 1, z_2, z_3, \ldots, z_k \geq 3, z_{k+1} \geq 2.$$
The number of such is
$$\binom{m-1-3(k-1)-2+k+1-1}{k+1-1} = \binom{m-2k}{k}.$$
Alternatively, we can represent a k-set by a sequence of k 1's and $m-k$ 0's in the usual way. Now we need every pair of 1's separated by at least 2 0's. We can start with a sequence of $m-2k$ 0's, choose k of them and replace each of these k 0's by 100. This process is reversible. For the 0,1 sequences we are counting each 1 is followed by at least 2 0's. Just replace 100 by 0 to get a sequence of $m-k$ 0's.

(b) Let $A = \{a, b, c\}^n$. Then let
$$A_k = \{x \in A : \ x_k = a, x_{k+1} = b, x_{k+2} = c\}$$

for $k = 1, 2, \ldots, n-2$.
Let $\mathcal{S} = \bigcup_{k \geq 0} \mathcal{S}_k$. Then

$$|A_S| = \begin{cases} 3^{n-3|S|} & S \in \mathcal{S}_{|S|} \\ 0 & S \notin \mathcal{S}_{|S|} \end{cases}.$$

Then we must compute

$$\begin{aligned} \left| \bigcap_{i=1}^{m} \bar{A}_i \right| &= \sum_{S \in \mathcal{S}} (-1)^{|S|} |A_S| \\ &= \sum_{S \in \mathcal{S}} (-1)^{|S|} 3^{n-3|S|} \\ &= 3^n \sum_{k=0}^{m} (-1)^k |\mathcal{S}_k| 3^{-3k} \\ &= 3^n \sum_{k=0}^{m} (-1)^k | \binom{m-2k}{k} 3^{-3k} \end{aligned}$$

3. Find an expression for the size of the set

$$\{(x_1, x_2 \ldots, x_m)\} \in Z^m : \ x_1 + x_2 + \cdots + x_m = n \ and \ a \leq x_j \leq b \ for \ j = 1, 2, \ldots, m\}.$$

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

Solution: Let

$$A = \{(x_1, x_2 \ldots, x_m)\} \in Z^m : \ x_1 + x_2 + \cdots + x_m = n \ and \ a \leq x_j \ for \ j = 1, 2, \ldots, m\}.$$

Then let

$$A_i = \{x \in A : \ x_i \geq b + 1\}.$$

Now,

$$\begin{aligned} A_S = \{(x_1, x_2 \ldots, x_m)\} \in Z^m : \ & x_1 + x_2 + \cdots + x_m = n \\ & and \ a \leq x_j \ for \ j \notin S, \ b + 1 \leq x_j \ for \ j \in S\}. \end{aligned}$$

So,

$$|A_S| = \binom{n - 1 - b|S| - (a-1)(m - |S|)}{m-1}.$$

Then we must compute

$$\begin{aligned} \left| \bigcap_{i=1}^{m} \bar{A}_i \right| &= \sum_{S \subseteq [m]} (-1)^{|S|} |A_S| \\ &= \sum_{S \subseteq [m]} (-1)^{|S|} \binom{n - 1 - b|S| - (a-1)(m - |S|)}{m-1} \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n - 1 - bk - (a-1)(m-k)}{m-1}. \end{aligned}$$