21-301 Combinatorics
Homework 1
Due: Friday, September 5

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

satisfy $x_1 \geq 3$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 6$ and $x_5 \geq 0$?

Solution Let

$$y_1 = x_1 - 3, \qquad y_2 = x_2 - 10, \qquad y_3 = x_3 + 3, \qquad y_4 = x_4 - 6, \qquad y_5 = x_5.$$

An integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 100$ such that $x_1 \geq 3$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 6$ and $x_5 \geq 0$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 + y_5 = 84$ such that $y_1, \ldots, y_5 \geq 0$. From a result in class,

$$|\{(y_1, y_2, y_3, y_4, y_5) : y_1, \ldots, y_5 \in Z_+ \text{ and } y_1 + \cdots + y_5 = 84\}| = \binom{84 + 5 - 1}{5 - 1} = \binom{88}{4}.$$

2. Prove the following equation:

$$\sum_{i=k}^{n} \binom{i}{k}\binom{n}{i} = \binom{n}{k} 2^{n-k}.$$

Solution Consider the set

$$\mathcal{S} = \{(A, B) : B \subseteq A \subseteq [n] \text{ and } |B| = k\}.$$

In words, $\mathcal{S}$ is the set of all ordered pairs consisting of a subset of $[n]$ and k elements of that set. We count the elements of $\mathcal{S}$ is two ways.

First we count with respect to the sets B. There are $\binom{n}{k}$ choices for B. Once B is fixed, $A \setminus B$ can be any subset of $[n] \setminus B$. There are 2^{n-k} such sets. Therefore, we have

$$|\mathcal{S}| = \binom{n}{k} n 2^{n-k}.$$

Now we count with respect to the set A. For $k, k+1, \ldots, n$ let

$$\mathcal{S}_i = \{(A, B) \in \mathcal{S} : |A| = i\}.$$

These sets form a partition of $\mathcal{S}$. There are $\binom{n}{i}$ choices for the set A in an ordered pair in $\mathcal{S}_i$. Once this set is fixed there are $\binom{i}{k}$ choices for B. Therefore

$$|\mathcal{S}_i| = \binom{n}{i}\binom{i}{k},$$

and

$$|\mathcal{S}| = \sum_{i=k}^{n} |\mathcal{S}_i| = \sum_{i=1}^{n} \binom{n}{i}\binom{i}{k}.$$

The result is given by noting that the two expressions for $|\mathcal{S}|$ are equal.

3. How many ways are there of placing k 1's and $n-k$ 0's at the vertices of an n vertex polygon, so that every pair of 1's are separated by at least 2 0's?

   Solution Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1,\ldots,a_k \geq 2$ such that $a_1+\cdots+a_k = n-k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

   Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of $a_1,a_2,\ldots,a_k$ that will now give π.

   There are $\binom{n-2k-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k}\binom{n-2k-1}{k-1}.$$