21-301 Combinatorics
Homework 10
Due: Monday, December 1

1. How many ways are there to k-color an $n \times n$ chessboard when n is odd. The group G is the usual 8 element group e, a, b, c, p, q, r, s.

   Solution: All we need do is compute the number of cycles in the permutations as applied to the chessboard. Let $n = 2m + 1$. In the table, $r \times s$ is short for r cycles of length s.

Permutation	Number of cycles
e	$n^2 \times 1$
a	$m(m+1) \times 4 + 1 \times 1$
b	$2m(m+1) \times 2 + 1 \times 1$
c	$m(m+1) \times 4 + 1 \times 1$
p	$mn \times 2 + n \times 1$
q	$mn \times 2 + n \times 1$
r	$mn \times 2 + n \times 1$
s	$mn \times 2 + n \times 1$

   Thus the number of colorings is

$$\frac{1}{8}(k^{n^2} + 2 \times k^{m(m+1)+1} + k^{2m(m+1)+1} + 4 \times k^{mn+n}).$$

2. How many ways are there to arrange 2 M's, 4 A's, 5 T's and 6 H's under the condition that any arrangement and its inverse are to be considered the same.

   Solution: The group G consists of $\{e, a\}$ where a is a reflection through the middle of the word. Now

$$|Fix(e)| = \frac{17!}{2!4!5!6!} = 85765680$$
$$|Fix(a)| = \frac{8!}{1!2!2!3!} = 1680$$

   A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter T. Then we arrange 1

M, 2 A's, 2 T"s and 3 H's in any order and then complete the sequence uniquely to a palindrome.

Thus by Burnside's theorem, the number of sequences is $\frac{85765680+1680}{2} =$ 42883680.

3. A necklace is made of 10 beads strung together in a cycle. Find the pattern inventory for the two colourings of the necklace when the group G is the dihedral group D_{10}.

   The *dihedral* group D_n is the group of symmetries of a regular n-gon under rotations $R_0, R_1, \ldots, R_{n-1}$ and reflections $S_1, S_1 \ldots, S_n$. Here, assuming n is even, the permutations are

   (i) $e = R_0, R_1, \ldots, R_{n-1}$ where R_i is *a rotation through* $i\pi/4$,

   (ii) $S_1, S_1, S_2, \ldots, S_{n/2}$ where S_i is *a rotation about an axis joining two opposite vertices*,

   (iii) $S_{n/2+1}, S_{n/2+2}, \ldots, S_n$ where $S_{n/2+i}$ is *a rotation about an axis joining the midpoints of two opposite edges.*

Solution: The group of symmetries consists of

(i) $e = r_0, r_1, \ldots, r_9$ where r_i is *rotation through* $i\pi/5$,

(ii) s_1, s_2, s_3, s_4, s_5 where s_i is *a rotation about an axis joining two opposite vertices*,

(iii) t_1, t_2, t_3, t_4, t_5 where t_i is *a rotation about an axis joining the midpoints of two opposite edges.*

π	e	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	s_i	t_i
ct(π)	x_1^{10}	x_{10}	x_5^2	x_{10}	x_5^2	x_2^5	x_5^2	x_{10}	x_5^2	x_{10}	$x_1^2x_2^4$	x_2^5

Thus

$$PI = \frac{1}{20}((b+w)^{10}+4(b^{10}+w^{10})+4(b^5+w^5)^2+5(b+w)^2(b^2+w^2)^4+6(b^2+w^2)^5).$$