9/3/08

$\{a, a, b, c, d, d\}$

$\cancel{\{a, b, c, d\}}$

$\{2 \times a,\ 1 \times b,\ 1 \times c,\ 2 \times d\}$

2 copies of a

1 copy of b

1 copy of c

2 copies of d

We have permutations of sets: $\{a, b, c, d\}$

e.g. $acdb$

We now consider permutations of multi-sets

$A = \{3 \times a,\ 4 \times b,\ 2 \times c\}$ $\qquad A^* = \{a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2\}$

a b b a c a b c b

How many?? $\qquad \dfrac{9!}{3!\,4!\,2!}$

Perm of A^*: $|A^*| = 9!$

$a_1\, b_3\, c_2\, c_1\, a_2\, b_4 \ldots$

$a_3\, b_2\, c_2\, c_1\, a_1\, b_3 \ldots$

$\# = 3!\,4!\,2!$

Perm of A

a b c c a b …

$$A = \{a_1 \times 1,\ a_2 \times 2, \ldots, a_n \times n\}$$

$$a_1 + a_2 + \cdots + a_n = m$$

$$\#\text{perms} = \frac{m!}{a_1! \cdots a_n!}$$

Multinomial Coefficients

$$\binom{m}{a_1, a_2, \dots, a_n} = \frac{m!}{a_1!\, a_2! \dots a_n!}$$

[Binomial: $n=2$, drop a_2 in the notation]

$$(x_1 + x_2 + \dots + x_n)^m = \sum_{\substack{a_1 + a_2 + \dots + a_n = m \\ a_1 \geq 0, \dots a_n \geq 0}} \underbrace{\binom{m}{a_1, a_2, \dots a_n} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}}$$

$(x_1 + \dots + x_n) \times (x_1 + \dots + x_n) \times (\quad) \times (\quad) \times \dots$

choose x_1 a_1 lines, x_2 a_2 lines, ...

Choose x_1 a_1 lines

x_2 a_2 lines

$\vdots$

Give me a permutation

$$\{a_1 \times x_1, a_2 \times x_2, a_3 \times x_3 \dots a_n \times x_n\}$$

$$\# = \binom{n}{a_1, \dots a_n}$$

Balls in boxes.

m balls $\quad 1, \; 2, \; 3, \; \ldots, \; m$

$$\Downarrow$$

$B_1 \quad B_2 \quad \ldots \quad B_n$

$$\# \text{ways} = n^m = \sum \binom{m}{b_1, b_2, \ldots b_n}$$

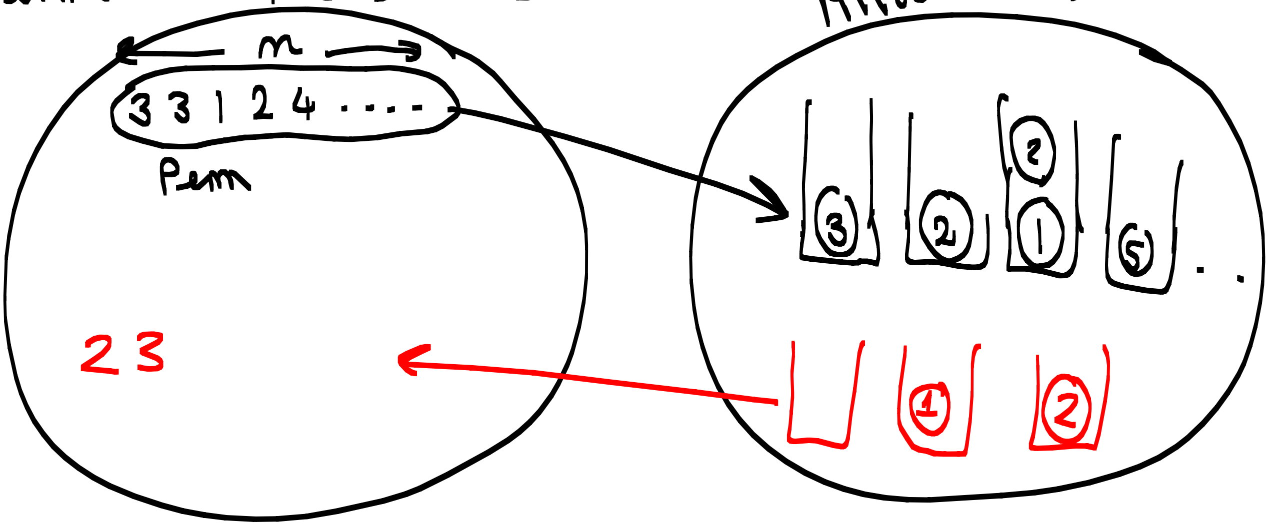
Follows from either (i) summing over allocations (ii) putting $x_i = 1 \; \forall i$ in multinomial thm)

How many have b_i balls in box i, $i = 1, 2, \ldots n$

$$\# \text{ways} = \binom{n}{b_1, b_2, \ldots b_n}$$

Perms of $\{b_1 \times 1, b_2 \times 2, \ldots\}$

Allocations, b_1 into Box 1, ..

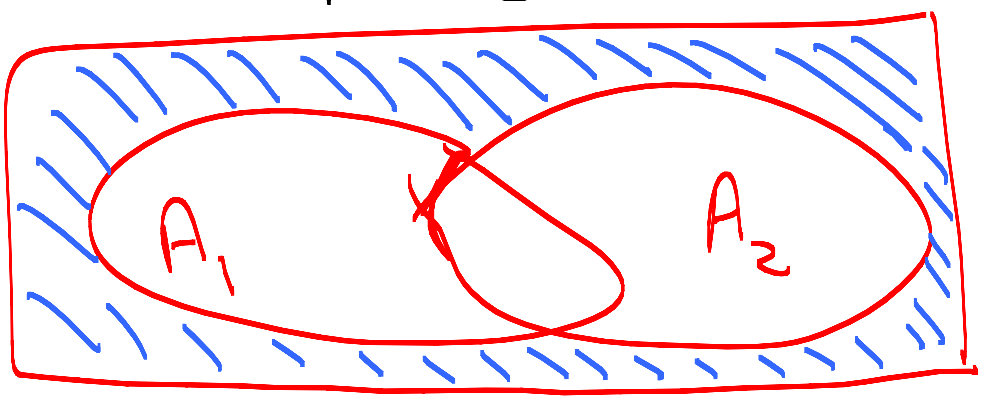


Inclusion-Exclusion

$A_1, A_2 \subseteq A$

2 sets

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$



$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |A| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

elements **not in** any of A_1, A_2, A_3.

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \cdots \cap \overline{A}_{10}| =$$

$$|A| \quad + \quad - \quad + |A_3 \cap A_7 \cap A_8 \cap A_9|$$

$$- |A_1 \cap A_3 \cap A_7 \cap A_9 \cap A_{10}|$$

$$+ \quad \cdot \quad - \quad -$$

$A_1, A_2, \dots, A_N$

For $S \subseteq \{1, 2, \dots, N\}$

$A_S = \bigcap_{i \in S} A_i$

$|A_1 \cap A_5 \cap A_8|$

$= A_{\{1,5,8\}}$

$$\left| \bigcap_{i=1}^{N} \overline{A}_i \right| = \sum_{S} (-1)^{|S|} |A_S|$$