9/26/08

Discrete Probability

Ω is a countable set.

$P : \Omega \; (R^+)$

$$\sum_{w \in \Omega} P(w) = 1$$

$P(w)$ = Probability of w.

Fair Coin:

$\Omega (H, T)$. $P(H) = P(T) = \frac{1}{2}$.

Dice

$\Omega = \{1, 2, \ldots, 6\}$ $P[i] = \frac{1}{8}$ $\forall i$

Geometric Distribution

Repeat an experiment until success.

$P(S) = p,$ $P(F) = 1-p$

$p_k = (1-p)^{k-1} p = P[k \text{ trials}]$

Two Dice

$\Omega = [6]^2$ $\qquad P[i,j] = \frac{1}{36}, \ \forall i,j$

$\Omega = \{2, 3, 4, \ldots, 12\}$

$P[2] = \frac{1}{36}$

$P[3] = \frac{2}{36}$ $\qquad (1,2)$ or $(2,1)$

etc

$$A \subseteq \Omega$$

Event

$$P[A] = \sum_{\omega \in \Omega} P(\omega).$$

Pennsylvania Lottery

Choose 7 numbers I from $[80]$

State choose 11 numbers J

WIN if $I \subseteq J$

$\text{WIN} = \{J : J \supseteq I\}$

$$P[\text{WIN}] = \frac{\binom{73}{4} \leftarrow \#J \supseteq I}{\binom{80}{11} \leftarrow \text{state choice}}$$

Booles Inequality

$A, B \subseteq \Omega$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\leq P(A) + P(B).$$

$$P[A_1 \cup A_2 \cup \dots \cup A_N] \leq$$

$$P[A_1] + P[A_2] + \dots + P[A_N]$$

Coloring Problem

Given sets $A_1, A_2, \ldots, A_n \subseteq A$

$|A_i| = k$ and $n < 2^{k-1}$.

Claim: $\exists$ partition $A = R \cup B$

s.t. $A_i \cap R \neq \emptyset \quad \forall i$

$A_i \cap B \neq \emptyset \quad \forall i$

$A_1, \ldots A_6 = \{1,2\}, \{1,3\}, \{1,4\},$

$\ldots \{3,4\}$

$k = 2$

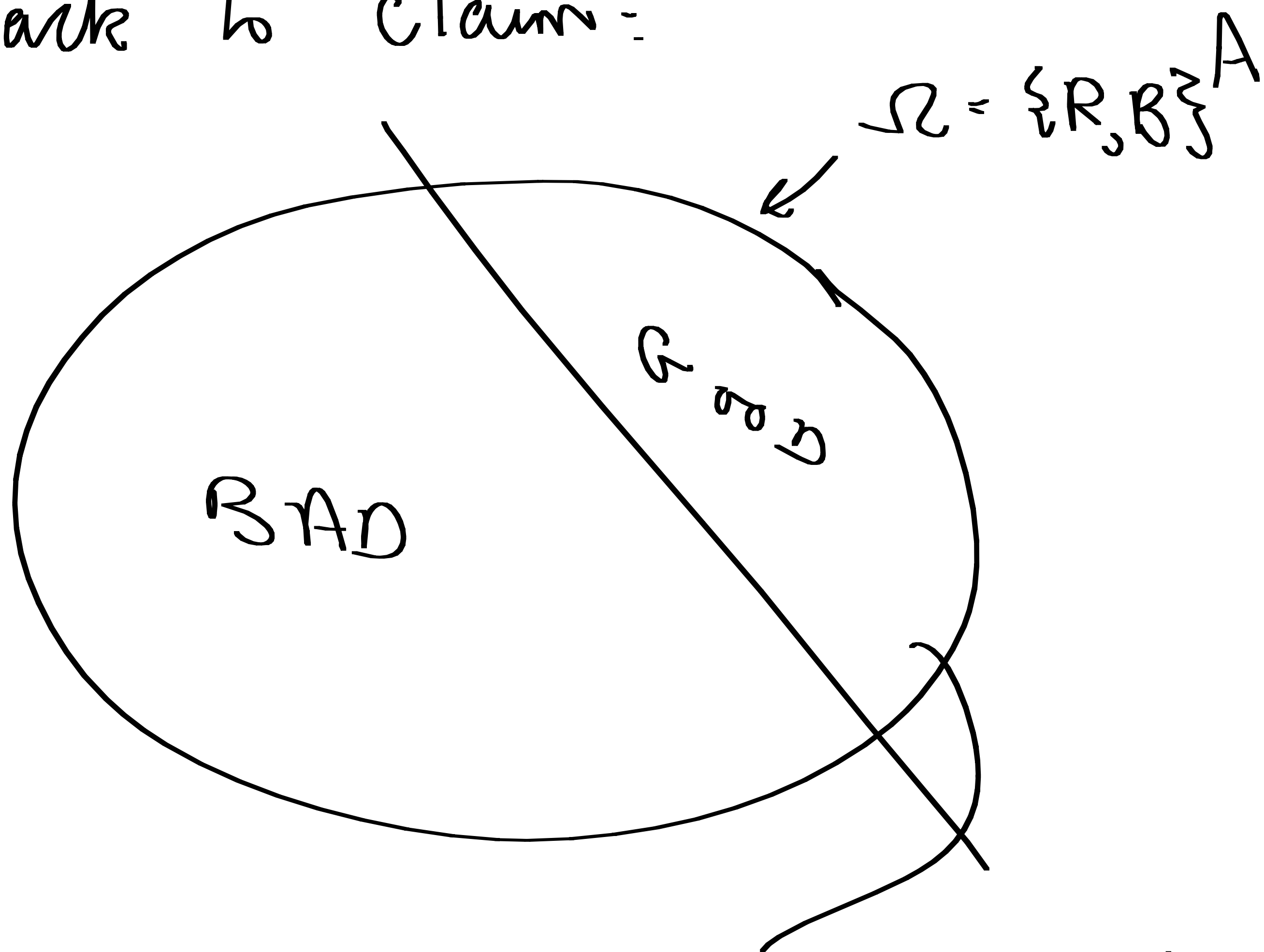
Impossible to color these.

Why?

Either $|R| \geq 2$ or $|B| \geq 2$

If $|R| \geq 2$ then any 2-subset of R does not meet B.

Back to claim:



$P[\text{GOOD}] > 0 \Rightarrow \text{GOOD} \neq \emptyset$

Color A randomly

$$BAD = \{ \exists i : A_i \subseteq R \text{ or } A_i \subseteq B \}$$

$$P[BAD] < 1 \Rightarrow P[GOOD] > 0.$$

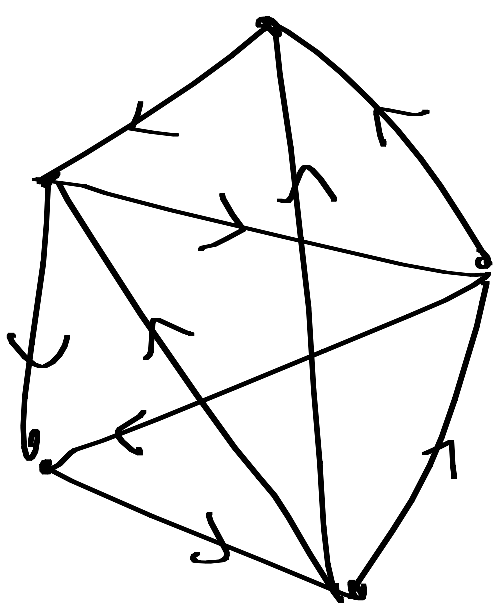
$$BAD = \bigcup_{i=1}^{n} BAD_i$$

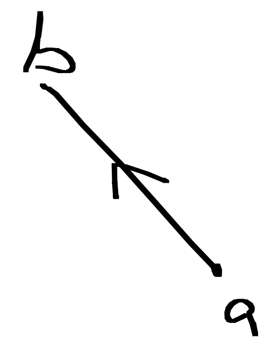
$$BAD_i = \{ A_i \subseteq R \text{ or } A_i \subseteq B \}$$

$$
\begin{aligned}
P(BAD) &= Pr\Big(\bigcup_{i=1}^{n} BAD_i\Big] \\
&\leq \sum_{i=1}^{n} Pr(BAD_i) \\
&= \sum_{i=1}^{n} \frac{1}{2^{k-1}} \\
&= \frac{n}{2^{k-1}} \\
&< 1 \quad \text{by assumption.}
\end{aligned}
$$

Tournaments

n players, each play each other.





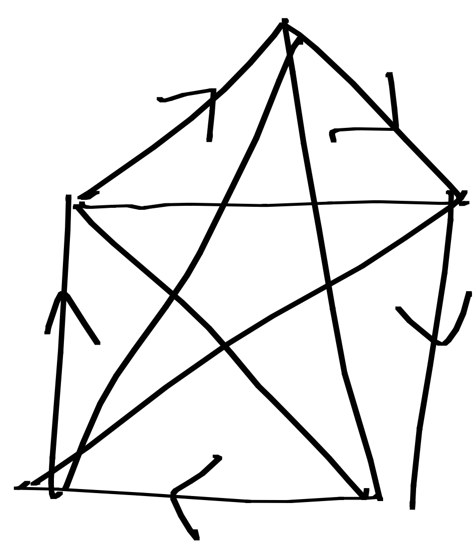
b beats a

Orientation of a complete graph.

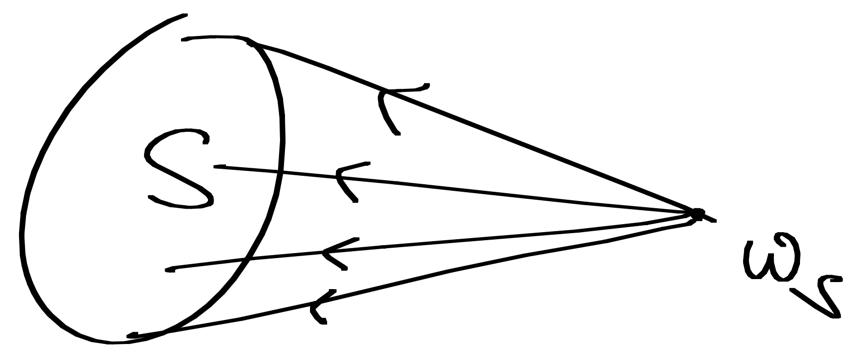
Fix some integer k.

Is it possible that for every set S, $|S| = k$ there is a person that beats S?

$k = 1$



Take a random tournament



$\mathcal{E}_S = \{$ no one beats all of $S\}$

$$\mathcal{E} = \bigcup_S \mathcal{E}_S$$

We show $\Pr(\mathcal{E}) < 1$.

$$\Pr(\mathcal{E}) = \Pr\left(\bigcup_S \mathcal{E}_S\right)$$

$$\leq \sum_S \Pr(\mathcal{E}_S)$$

$$= \sum_S \left(1 - \left(\tfrac{1}{2}\right)^k\right)^{n-k}$$

S

$\Pr(1 \text{ does not beat } S) = 1 - \left(\tfrac{1}{2}\right)^k$

$$P(\mathcal{E}) \leq \sum_{|S|=k} \left(1 - \left(\tfrac{1}{2}\right)^k\right)^{n-k}$$

$$= \binom{n}{k} \left(1 - \left(\tfrac{1}{2}\right)^k\right)^{n-k}$$

$$< n^k \left(1 - \left(\tfrac{1}{2}\right)^k\right)^{n-k}$$

k is fixed as n grows

$\to 0$

$$P(S) \le n^k \left(1 - \left(\tfrac{1}{2}\right)^k\right)^{n-k}$$

$$\le n^k \exp\left\{-(n-k) \cdot \left(\tfrac{1}{2}\right)^k\right\}$$

Use $1 + x \le e^x \quad \forall x.$

Take logarithm

$$k \log n - (n-k) * \left(\tfrac{1}{2}\right)^k$$