9\24\08

Derangements:

d_n = # of derangements of $\{1, 2, \ldots, n\}$

choose i s.t. $\pi(i) = i$

derange rest

$$n! = \sum_{k=0}^{n} \underbrace{\binom{n}{k} d_{n-k}}_{\text{\# of permutations with exactly } k \text{ fixed points}}$$

permutations

$[\pi(i) = i]$ Fixed pt.

$$1 = \sum_{k=0}^{n} \underbrace{\frac{1}{k!}}_{a_k} \cdot \underbrace{\frac{d_{n-k}}{(n-k)!}}_{b_{n-k}}$$

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$a_m = \frac{1}{m!}$ $\qquad a(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$

$b_m = \frac{d_m}{m!}$ $\qquad b(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m$

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

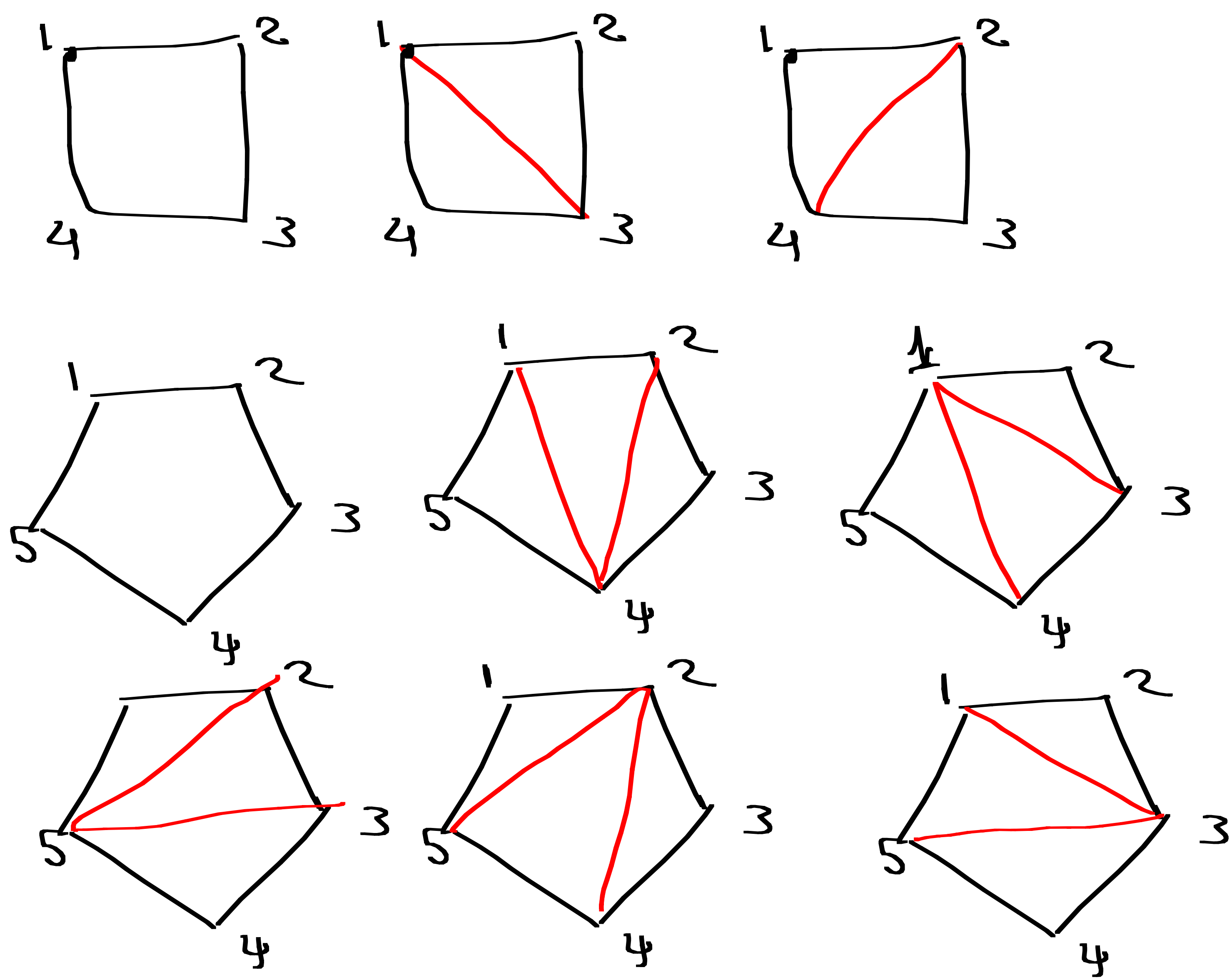
$$\frac{1}{1-x} = a(x)\, b(x)$$

$$= e^{x}\, b(x)$$

$$b(x) = e^{-x} \times \frac{1}{1-x}$$

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \times 1$$

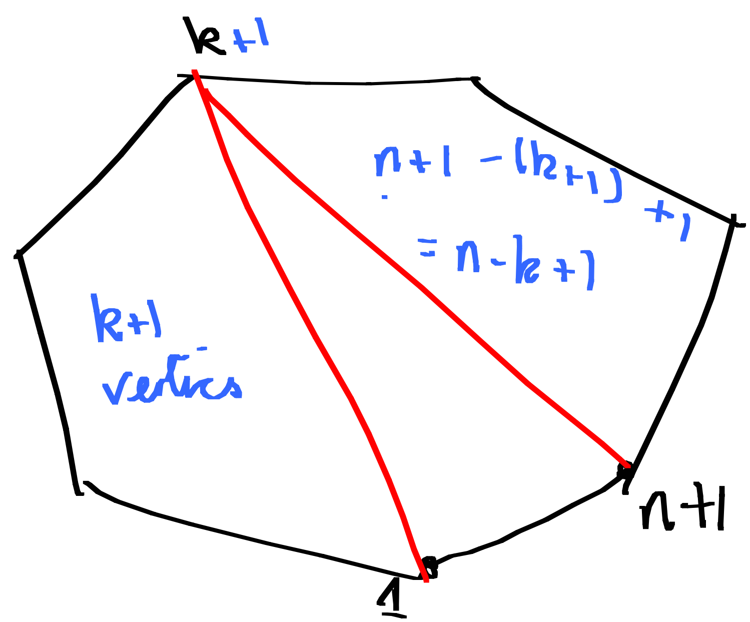
Triangulations of a polygon



a_n = # of triangulations of P_{n+1}

$a_0 = 0, \quad a_1 = a_2 = 1$

$$a_n = \sum_{k=0}^{n} a_k \, a_{n-k}$$



$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$$

$$a(x) - a_0 - a_1 x$$

$$= a(x) - x$$

$\underline{n=1}$ $\quad \sum_{k=0}^{1} a_k a_{n-k} = 0$

$n=0$ $\quad \sum_{k=0}^{0} a_k a_{n-k} = 0$

$$a(x)^2 - a(x) + x = 0$$

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \quad \text{or} \quad a(x) = \frac{1-\sqrt{1-4x}}{2} \checkmark$$

$$a(0) = a_0 = 0$$

$$a(x) = \frac{1}{2}\left(1 - (1-4x)^{1/2}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} (-4)^n x^n$$

$$a(x) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} (-4)^n x^n$$

$$a_0 = 0$$

$$a_n = -\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\cdots\left(\frac{1}{2}-n+1\right)}{2n!} (-4)^n$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\cdots\left(n-\frac{3}{2}\right)}{2\,n!} \cdot 4^n$$

$$= \frac{1.3.5.\cdots\cdots(2n-3)}{n!} 2^{n-1}$$

$$a_n = \frac{1 \cdot {}^{2} 3 \cdot {}^{4} 5 \cdot {}^{6} \cdots (2n-3)^{2n-2}}{2 \cdot 4 \cdot 6 \cdots (2n-2)\; n!} \, 2^{n-1}$$

$$= \frac{(2n-2)!\; 2^{n-1}}{2^{n-1}\,(n-1)!\; n!}$$

$$= \frac{(2n-2)!}{n\,(n-1)!^2}$$

$$= \frac{1}{n}\binom{2n-2}{n-1}.$$