9\10\08

Totient Function:

$\phi(n) = \# \, x \leq n$ s.t. $x$ & $n$ are co-prime. [only common factor is 1]

$\phi(12) = 4$ $\quad$ 1, 5, 7, 11

$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorisation.

$$A = [n]$$

$$A_i = \{ x \leq n : p_i \mid x \}$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A_i} \right|$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} |A_S|$$

$$|A_1| = \frac{n}{p_1} \quad ; \quad |A_{\{1,2\}}| = \frac{n}{p_1 p_2}$$

$$x : p_1 \mid x$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i}$$

$$\phi(n) = \sum_{S} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

# Scrambled Allocations

$n$ boxes $\quad B_1, B_2, \ldots, B_n$

$2n$ balls $\quad b_1, b_2, \ldots, b_{2n}$

Balls $\longrightarrow$ Boxes

(i) Each box gets two balls

(ii) Box $i$ does not get $b_{2i-1}, b_{2i}$

$A_i = \{\text{allocations, } B_i \text{ contains } b_{2i-1}, b_{2i}\}$

# scrambled allocations

$$= \sum_{S} (-1)^{|S|} |A_S|$$

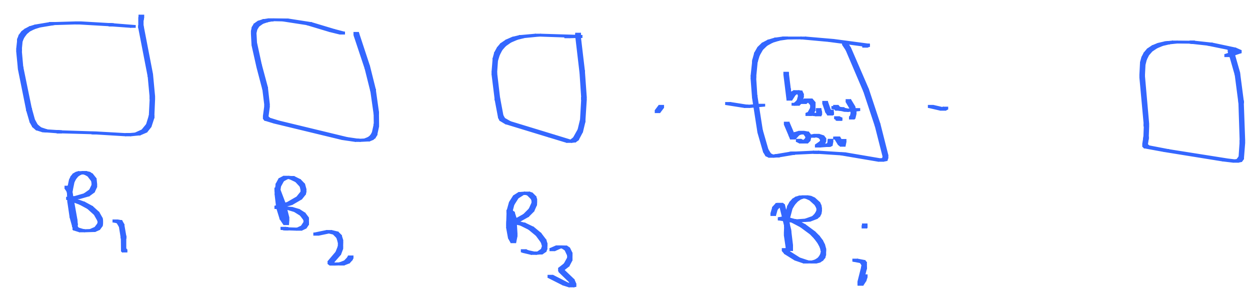
$$|A_\emptyset| = \underbrace{\binom{2n}{2}}_{B_1} \underbrace{\binom{2n-2}{2}}_{B_2} \cdots$$

$$= \frac{(2n)!}{2^n}$$

In general

$$|A_S| = \frac{[2(n-|S|)]!}{2^{n-|S|}}$$

$i \in S$




$2(n-|S|)$ balls

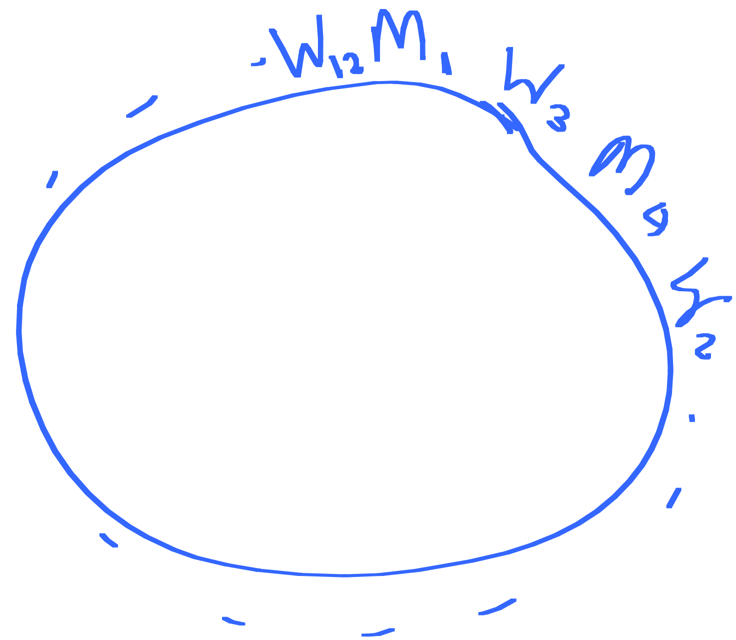
$\Downarrow$

$n-|S|$ boxes

# Problème des Ménages

$n$ male-female couples.

$M_1 W_1 \; M_2 W_2 \; \cdots$



$M_n$ = # of feasible arrangements.

$A_i = \{$ seatings s.t. couple $i$ sit together $\}$

Use inclusion-exclusion

If $|S| = k$ then

$$|A_S| = 2 \, k! \, ((n-k)!)^2 \times d_k$$

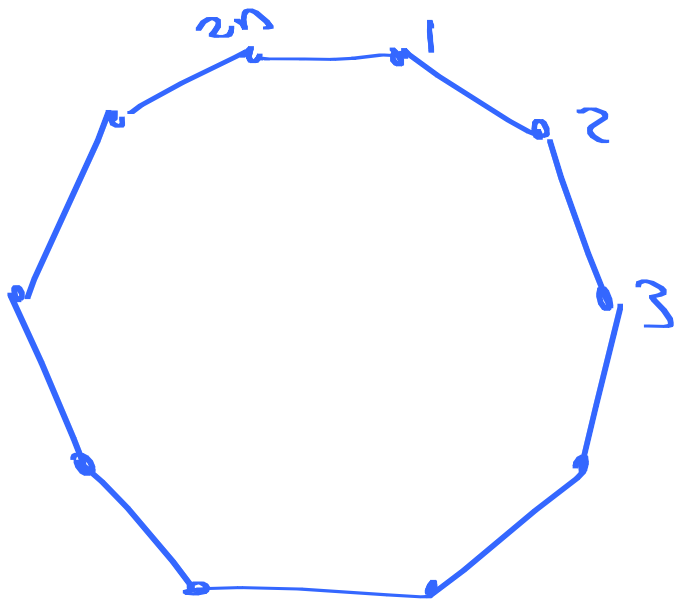
Seat 1st as M or W

allocate k couples to k places

putting in rest

# of places for k couples

$$d_k = \frac{2n}{k}\binom{2n-k-1}{k-1}$$




$1 \}$ couple

Place $k$ 1's, each separated by 0.