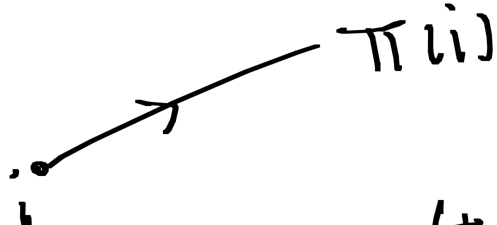
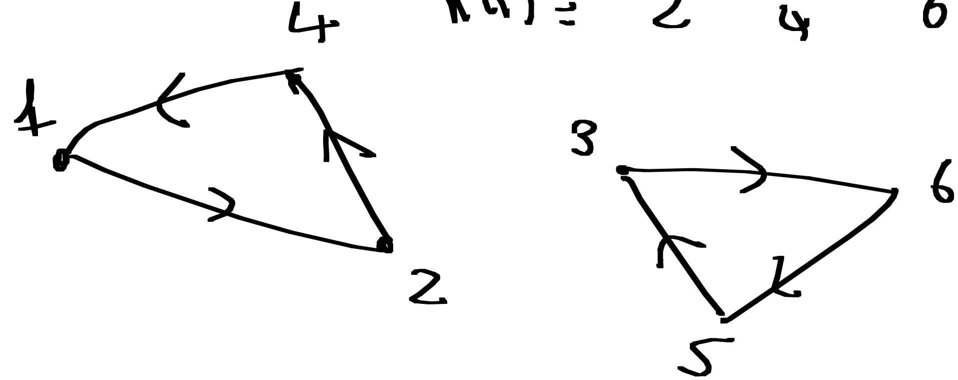
9\08\08

How many permutations on $1, 2, \cdots, n$ are such that there do not exist $i \neq j$ such that $\pi(i) = j$ and $\pi(j) = i$?

Given permutation π, there is digraph D_π

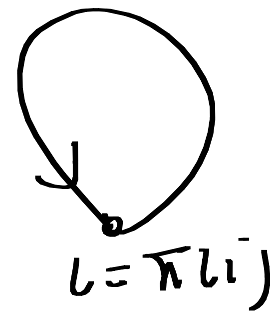


$i =$	1	2	3	4	5	6
$\pi(i) =$	2	4	6	1	3	5

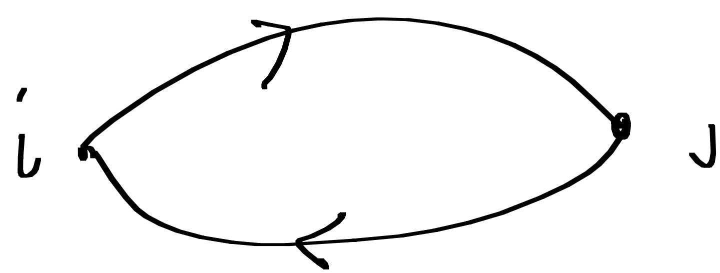


Derangement:

D_π does not contain a loop



What we are asking now is how many π have no



$A = \{\text{permutations}\}$

$N = \binom{n}{2}$

$B_{i,j} = \{\pi : \pi(i) = j \wedge \pi(j) = i\}$

We want to know

$$\left| \bigcap_{i,j} \overline{B_{i,j}} \right|$$

$\underline{n = 4}$

$$\begin{array}{cccccc} B_{1,2} & B_{1,3} & B_{1,4} & B_{2,3} & B_{2,4} & B_{3,4} \\ \uparrow & & & & & \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array}$$

In general

$$A_1, A_2, \ \cdots \ A_{\binom{n}{2}}$$

$$\underset{A_{\{1,6\}}}{\underbrace{B_{1,2} \cap B_{3,4}}} = \left\{ \pi : \begin{array}{l} \pi(1) = 2 \ \wedge \ \pi(2) = 1 \\ \wedge \ \pi(3) = 4 \ \wedge \ \pi(4) = 3 \end{array} \right\}$$

$$|A_\emptyset| = n!$$

$$|A_1| = |B_{1,2}| = (n-2)! \qquad \begin{array}{ccccccc} 2 & 1 & - & - & - & - & - \\ \times & \times & \times & \times & \times & \times & \times \end{array}$$

$$|A_S| = (n-2)! \text{ if } |S| = 1$$

$$|A_{\{1,2\}}| = |A_1 \cap A_2| = 0$$

$$B_{1,2} \cap B_{1,3}$$

$$|B_{1,2} \cap B_{3,4}| = \qquad \begin{array}{ccccccc} 2 & 1 & 4 & 3 & - & - & - \\ \times & \times & \times & \times & \times & \times & \times \end{array}$$

$$|A_S| = \begin{cases} 0 \\ (n-4)! \end{cases} \quad |S| = 2$$

Start of inclusion-exclusion formula

$$\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$$

$$n! - \binom{n}{2}\cdot(n-2)! + \cancel{??}\,(n-4)!$$

choose 2 disjoint pairs

$$\frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \frac{1}{2}\binom{n}{2,2,n-4}$$

$i,j \quad k,l$

$1,2 \quad 3,4$

$$|A_S| = \begin{cases} 0 \\ (n-2k)! \end{cases}$$

$|S| = k$

How many

$k = 3$

$$\frac{1}{3!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2} = \frac{1}{3!}\binom{n}{2,2,2,\,n-6}$$

$$\# = \frac{1}{k!}\binom{n}{\underbrace{2,2,2,\dots,2}_{k},\,n-2k}$$

$$\#\text{perms} = \sum_{k=0}^{n/2} (-1)^k \binom{n}{2,2,\dots,2,n-2k} (n-2k)!$$

$$= \sum_{k=0}^{n/2} (-1)^k \frac{n!}{2^k}$$

$$\longrightarrow e^{-1/2} \quad \text{as } n \to \infty$$

Onto-Functions (Surjections)

$$[n] \xrightarrow{\text{onto}} [m]$$

$$A = \{ f : [n] \longrightarrow [m] \}$$

$$|A| = m^n .$$

$$\# \text{surjections} = \left| \bigcap_{i=1}^{?} \overline{A_i} \right|$$

f is a surjection if $\forall y \in [m]$

there is an $x \in [n]$ with $f(x) = y$

Every y is covered, or every is not (not covered)

turn this into

f does not have a certain property

for $y = 1, 2, \ldots$

$A_y = \{ f : f(x) \neq y \ \forall x \}$

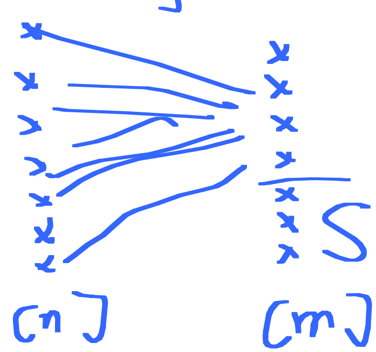
$$\# \text{surjections} = \left| \bigcap_{y \in [m]} \bar{A}_y \right|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} \, |A_S|$$

??

$$A_S = \{ f : f(x) \cancel{\in} S, \ \forall x \in [n] \}$$

A_S



$$|A_S| = (m - |S|)^n$$