9/05/08

$A_1, A_2, \ldots, A_N \subseteq A$

$$\left| \bigcap_{i=1}^{N} \overline{A_i} \right| = \underbrace{\sum_{S \subseteq [N]} (-1)^{|S|} |A_S|}_{2^n \text{ terms}}$$

\# of elements not in any of the $A_i$'s.

How many integers in 10,000 are not divisible by 3, 7, 14.

$A = [10{,}000]$

$A_1 = \{ n : n \text{ is divisible by } 3 \}$

$A_2 = \{ \text{" - - — — -} \quad 7 \}$

$A_3 = \{ \text{- - — —} \quad 14 \}$

# we want $= |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3|$

$= \underset{10000}{|A|}$

$- \underset{3333}{|A_1|} - \underset{1428}{|A_2|} - \underset{714}{|A_3|}$

$+ \underset{476}{|A_{\{1,2\}}|} + \underset{238}{|A_{\{1,3\}}|} + \underset{714}{|A_{\{2,3\}}|}$

$- \underset{238}{|A_{\{1,2,3\}}|}$. $=$

# Derangements

A derangement of $[n] = \{1, 2, \dots, n\}$ is a permutation $\pi$ s.t.

$$\pi(i) \neq i \quad \forall i$$

$n = 6$

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\pi(i)$ | 4 | 3 | 5 | 1 | 6 | 2 |

$\pi(i) \neq i$ $\quad \forall i$

$\bigcap_i \overline{A}_i$

Not in the set of $\pi$
such that $\pi(i) = i$

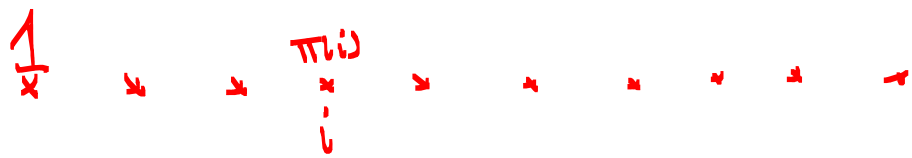
$A_i = \{\pi : \pi(i) = i\}$

We want $\pi \notin A_i \quad \forall i$

$|A_S| = ?$

$S = \emptyset$ : $A_\emptyset = A$ $\quad |A| = n!$

$S = \{1\}$ $\quad A_{\{1\}} = \{\pi : \pi(1) = 1\}$




$|A_1| = |A_2| = \cdots = |A_n| = (n-1)!$

$S = \{1,2\}$

$A_S$

$1 \quad 2 \quad \cdot \quad \cdot \quad \pi(i) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$i$

$$|A_S| = (n-2)! \qquad S = \{1,2\}$$

$A_S$

$\pi(i) = i$

$i \in S$

$S$ positions are filled out.

$$|A_S| = (n - |S|)!$$

$N = n$

# derangements =

$$\sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)!$$

$$= \sum_{k=0}^{n} \sum_{|S|=k} (-1)^{k} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^{k} (n-k)! \times \#(S : |S| = k)$$

$$= \sum_{k=0}^{n} (-1)^k (n-k)! \binom{n}{k}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \sim n!\, e^{-1}$$

if $n$ is large

Proof of I-E formula.

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$x \in \bigcap_{i=1}^{N} \overline{A_i}$ iff $(1 \ \theta_{x,1})(1-\theta_{x,2}) \cdots (1-\theta_{x,N}) = 1$

$$\left|\bigcap_{i=1}^{N} \overline{A_i}\right| = \sum_{x \in A} \prod_{i=1}^{N} (1-\theta_{x,i})$$

$$= \sum_{x \in A} \sum_{S} (-1)^{|S|} \prod_{i \in S} \theta_{x,i}$$

(where I choose the $\theta$)

$$= \sum_{x \in A} \sum_{S} (-1)^{|S|} \prod_{i \in S} \theta_{x,i}.$$

$$= \sum_{S} (-1)^{|S|} \sum_{x \in A} \underbrace{\prod_{i \in S} \theta_{x,i}}_{1 \text{ iff } x \in A_S}$$

$$= \sum_{S} (-1)^{|S|} |A_S|.$$