8\29\2008

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \quad (1)$$

<u>Proof</u>

(1) Induction on n.

Hypothesis $H(n)$: (1) is true $\forall k \le n$.

$n = 1$: $H(1)$ is true?

$$\binom{k}{k} + \cdots + \binom{1}{k} = \binom{2}{k+1} \quad \forall k \le 1$$

$k = 0$: $\binom{0}{0} + \binom{1}{0} = \binom{2}{1}$ $1+1=2$; $k=1$ $\binom{1}{1} = \binom{2}{2}$.

Suppose $H(n)$ is true, $\forall k \le n$

Consider $k \le n+1$

$$\underbrace{\boxed{\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}} + \binom{n+1}{k}}_{\text{LHS}} =$$

$$\binom{n+1}{k+1} + \binom{n+1}{k}$$

Induction hypothesis.

Pascal triangle

$$= \binom{n+2}{k+1}.$$

Second Proof

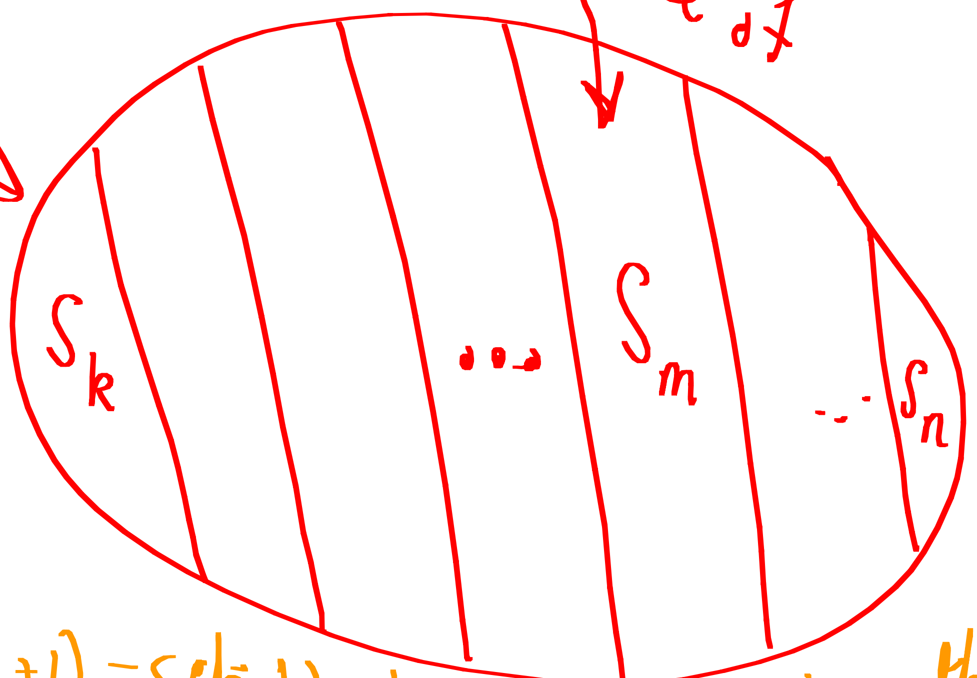
of (k+1)-sets containing n+1

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{m}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

size of

of (k+1) subsets of [n+1]

$$\binom{[n+1]}{k+1}$$

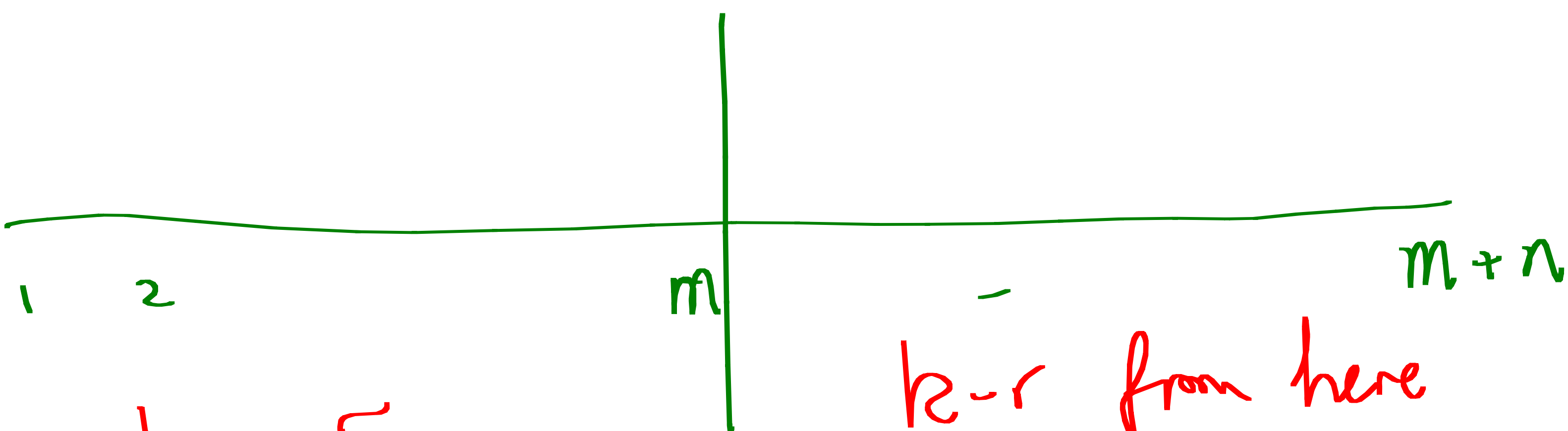


$S_m = \{$ (k+1)-sets that contain m+1 as their largest element $\}$

Vandermonde's Indentity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$$

\# of ways of choosing k elements



choose r from here

k-r from here

$r = 0, 1, \ldots, k$

Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Proofs: (i) Induction, using Pascal Triangle.

(ii)

$$\overbrace{(1+x)(1+x)(1+x)\cdots\cdots(1+x)}^{n}$$

Multiplying out gives 2^n terms, which I re-arrange

$(1+x)(1+x) = 1 + x + x + x^2 = 1+2x+x^2$

I get x^r when I choose x, r times.

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^n .$$

(i) $x = 1$

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

(ii) $x = -1$

$$0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots$$

$= \#$ even sets $- \#$ odd sets

$$(1+2)^n = 3^n = \underbrace{\sum_{r=0}^{n} \binom{n}{r} 2^r}_{\text{RHS counts ??}}$$

$$= \# \text{pairs } (A, B)$$

where $[n] \supseteq A \supseteq B$.

3^n : $\times \quad \times \quad \times \quad \underset{\times}{x} \quad \times \quad \times \quad \times \quad \times \quad \times$

$0, \; 1, \; 2$

$x \notin A \quad x \in A \setminus B \quad x \in B$

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$n(1+x)^{n-1} = \sum_{r=0}^{n} r \binom{n}{r} x^{r-1}$$

$x = \underline{1}$

$$n2^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r$$

ways of choosing a "committee"
+ a chair.