8/27/08

## Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

"n choose k"

$X$ is a finite set; $\binom{X}{k} = \{S \subseteq X : |S| = k\}$

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}$$

$|X| = n$. There are exactly $\binom{n}{k}$ $k$-subsets.

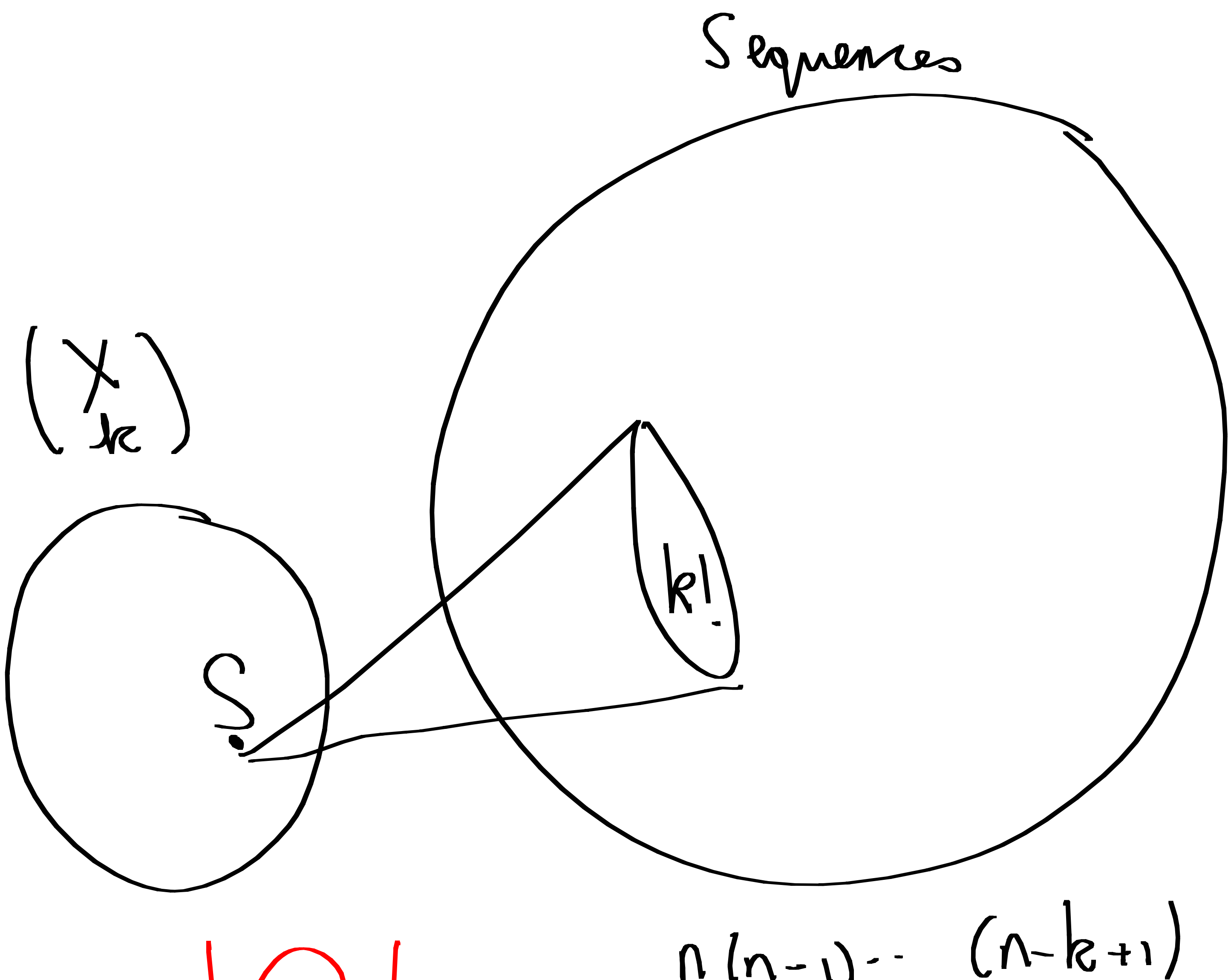
$\#$ of sequences of length $k$

$x_1, x_2, \ldots, x_k \leftarrow$ distinct

$= n(n-1)\cdots(n-k+1) = k! \left|\binom{X}{k}\right|$

$k!$ sequences $\longleftarrow S \in \binom{X}{k}$

$S \neq S'$ give rise to different sequences.




$$\text{Big}\,|O| = k! \times |O|$$

Number of solutions to

$$x_1 + x_2 + \cdots + x_n = m$$

$x_i \geq 0$ & integer

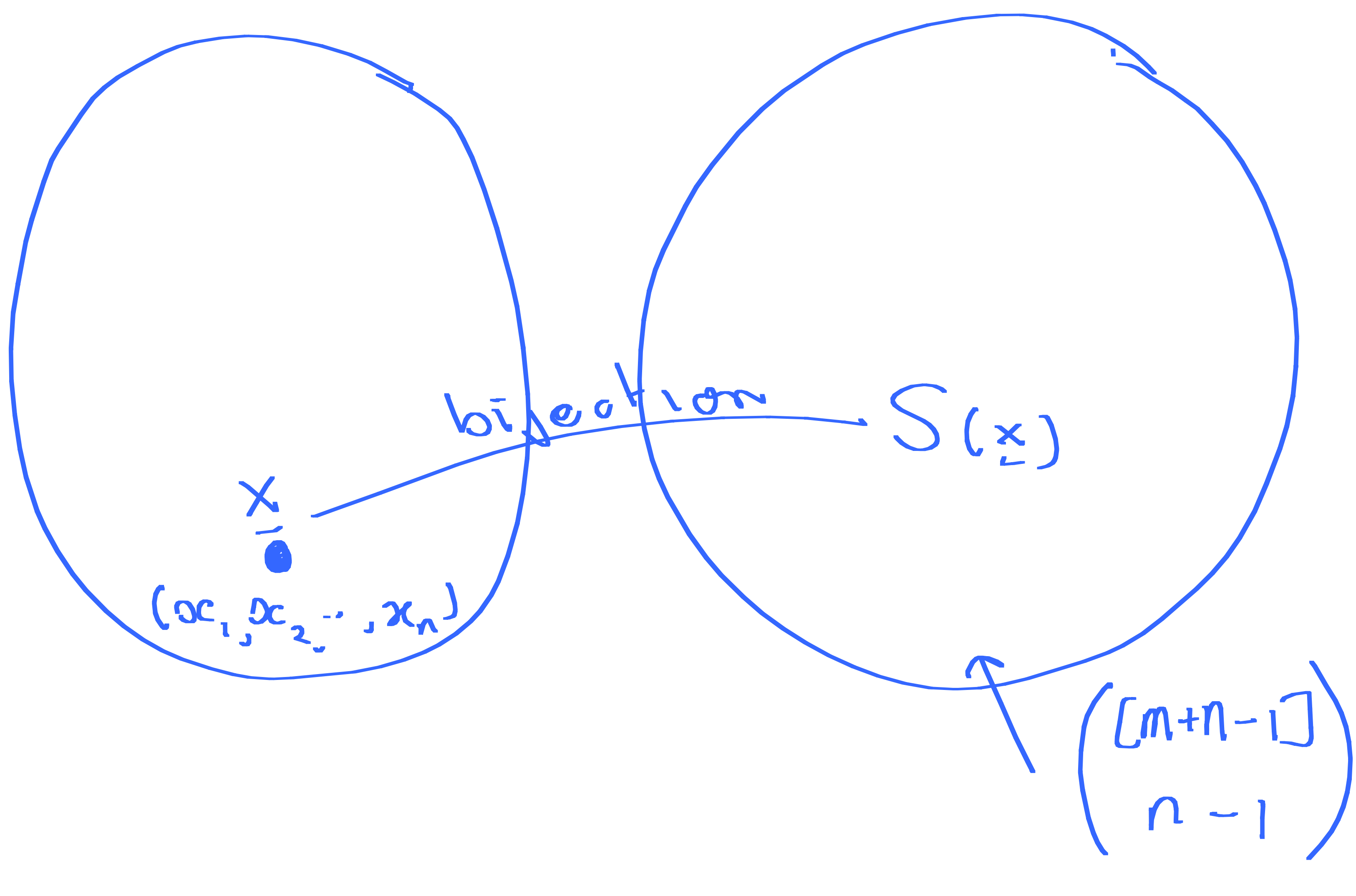
Ex: $m = 6$, $n = 3$

6, 0, 0    4, 2, 0

5, 1, 0    4, 0, 2    ....

5, 0, 1    4, 1, 1    0, 0, 6

Claim: $\# = \binom{m+n-1}{n-1}$



$\underline{X} = x_1, x_2, \dots, x_n$

$x_1 = 3, \; x_2 = 2, \; x_3 = 0$

$$\underbrace{\overbrace{* \; * \; *}^{x_1} \; \otimes \; \overbrace{* \; *}^{x_2} \; \otimes \; \otimes \; \cdots \; * \; * \; \otimes \; \overbrace{* \; *}^{x_n}}_{m+n-1}$$

$n-1$

$\# = |S(m,n)|$

Suppose now

$$\boxed{\# \underline{y} = \binom{m-n+n-1}{n-1} = \binom{m-1}{n-1}}$$

$$x_1 + x_2 + \cdots + x_n = m$$

and $x_1 \geq 1,\ x_2 \geq 1,\ \ldots,\ x_n \geq 1,$

$$S^*(m, n)$$

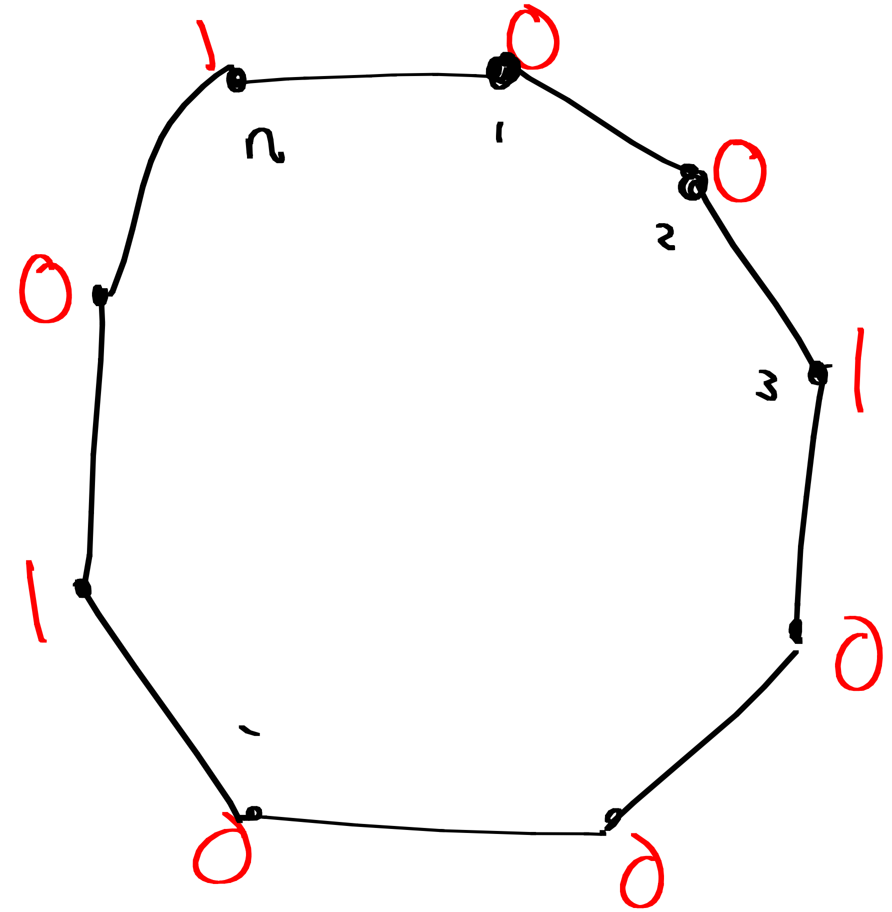
$x_j = y_j + 1$

$y_j = x_j - 1$

$$(y_1+1) + (y_2+1) + \cdots + (y_n+1) = m$$

$$y_1 + y_2 + \cdots + y_n = m-n$$

$$y_1 \geq 0,\ y_2 \geq 0,\ \cdots\ y_n \geq 0$$




$n=9$, $k=3$

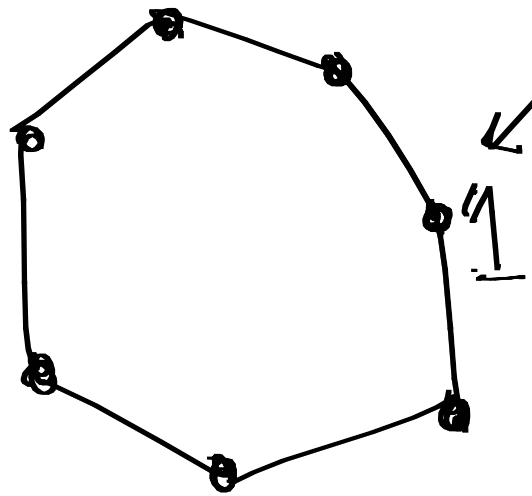
Fixed polygon

$n$ vertices

$k$ 1's and $(n-k)$ 0's to be placed at vertices.

No restriction: # ways $= \binom{n}{k}$

But now no adjacent 1's



Special vertex

$n$ choices

$\times \binom{n-k-1}{k-1}$

$$1 \underbrace{0,0,0,0}_{a_1} 1 \underbrace{0,0,0,0}_{a_2}, \dots, 0, \dots, \underbrace{0,0}_{a_k} 1$$

Each pattern appear $k$ times this way.

Total

$$\boxed{\frac{n}{k}\binom{n-k-1}{k-1}}$$

$a_1 + a_2 + \dots + a_k = n-k$

$a_1 \geq 1,\ a_2 \geq 1,\ \dots,\ a_k \geq 1$

# choices for $a_1, \dots, a_k$ $= \binom{n-k-1}{k-1}$

Properties of binomial coefficients:

(i) $\binom{n}{r} = \binom{n}{n-r}$

(ii) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

\# (k+1) sets in [n+1] that contain n+1 + # (k+1) sets in [n] = # (k+1) sets in [n+1]