8/25/08

How many sequence $a_1 a_2 \cdots a_m$ where $a_i \in \{1, 2, \ldots, n\} \leftarrow [n]$

$$\# = \underbrace{n \times n \times n \times \cdots \times n}_{m \text{ times}} = n^m$$

$= \#$ functions from $[m] \longrightarrow [n]$.

How many 1-1 functions from $[m] \to [n]$

$$\equiv \begin{array}{cccc} a_1 & a_2 & \cdots & a_m \\ \uparrow & \uparrow & \cdots & \uparrow \\ f(1) & f(2) & \cdots & f(m) \\ \uparrow & \nwarrow & & \nwarrow \\ n \text{ choices} & \begin{array}{c} n-1 \\ \text{choices} \end{array} & & \begin{array}{c} n-m+1 \\ \text{choices} \end{array} \end{array}$$

$\#$ 1-1 functions $[m] \to [n] = n(n-1) \cdots (n-m+1)$.

$\psi(n) = \#$ subsets of $[n]$

$= 2^n$

Proof

(1) By induction:

$\psi(0) = \underline{1}$ # subsets of empty set $\emptyset = \underline{1}$

$\psi(1) = 2$ subsets of $[1]$ are $\emptyset$, $[1]$.

$\psi(n+1) = \#\{\text{containing } n+1\} + \#\{\neg \text{ containing } n+1\}$

#subsets of $[n+1]$

$2^n + 2^n$

2^{n+1}

subsets of $[n]$

$A \xrightarrow{\text{BIJECTION}} f_A$

# of functions from $[n] \to \{0,1\}$

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

2^n

Mapping is a bijection:

(i) 1-1: $f_A = f_B \Rightarrow A = B$ or $A \neq B \Rightarrow f_A \neq f_B$

$A \neq B$, $\exists\, x \in A \setminus B$ or $x \in B \setminus A$ or both

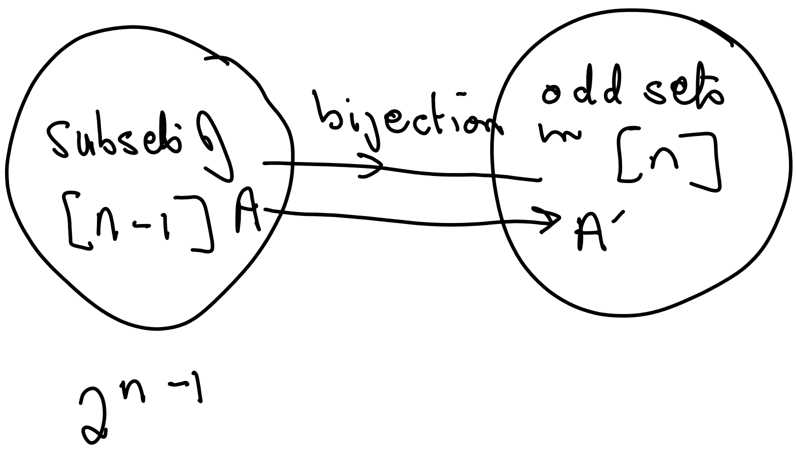
$f_A(x) = 1 \neq 0 = f_B(x)$.

(ii) Onto: $g : [n] \longrightarrow \{0,1\}$

$g = f_A$ where $A = \{x : g(x) = 1\}$

odd subsets of $[n] = 2^{n-1}$

$=$ # even subsets.



n is odd

$A' = A \quad |A|$ odd

$A' = A \cup \{n\} \quad |A|$ even.

Bijection

(i) $A' = B' \Rightarrow A = B$

$n \notin A'$: $A' = A = B' = B$ ✓

$n \in A'$ $A = A' \setminus \{n\}$

(ii) $\forall$ odd X, $\exists A$ s.t. $A' = X$.

$n \notin X$, $A = X$

$n \in X$ $A = X \setminus \{n\}$