11\19\08

X, group G G "acts" on X

$g * x \in X$

permutation

$O_x = \{ g * x : g \in X \}$

Orbits partition X

How many orbits

Lemma

$$|O_x|\,|S_x| = |G|$$

Stabiliser of x

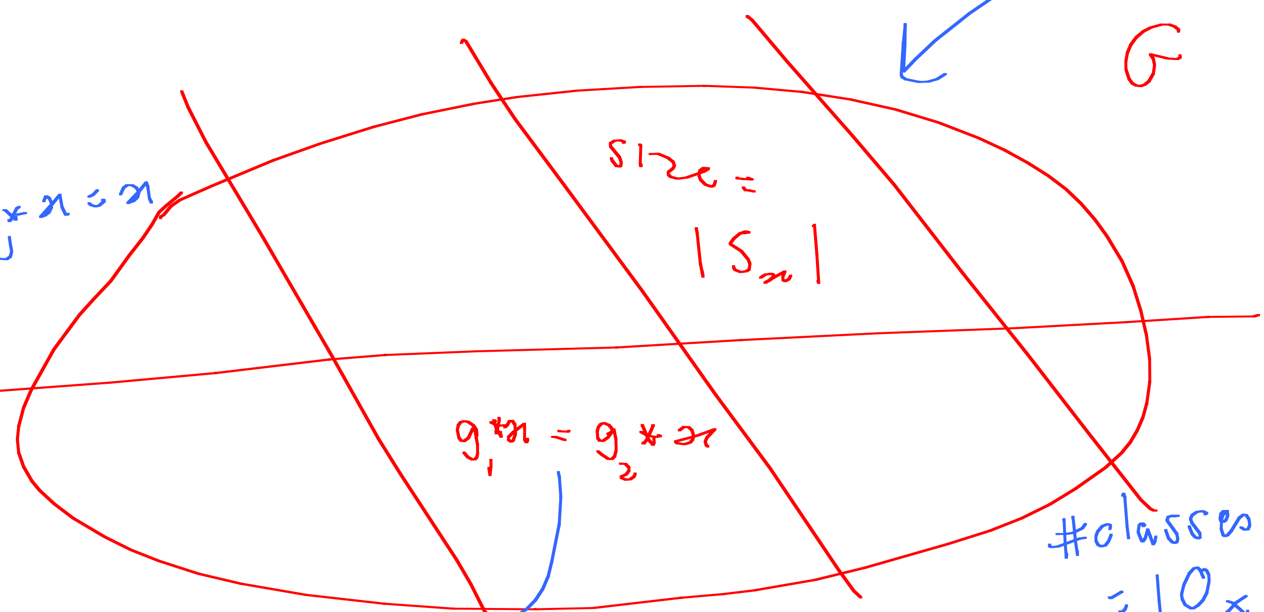
$$S_x = \{ g : g * x = x \}$$

$g_1 \in g_2 \circ S_x$

Fix x

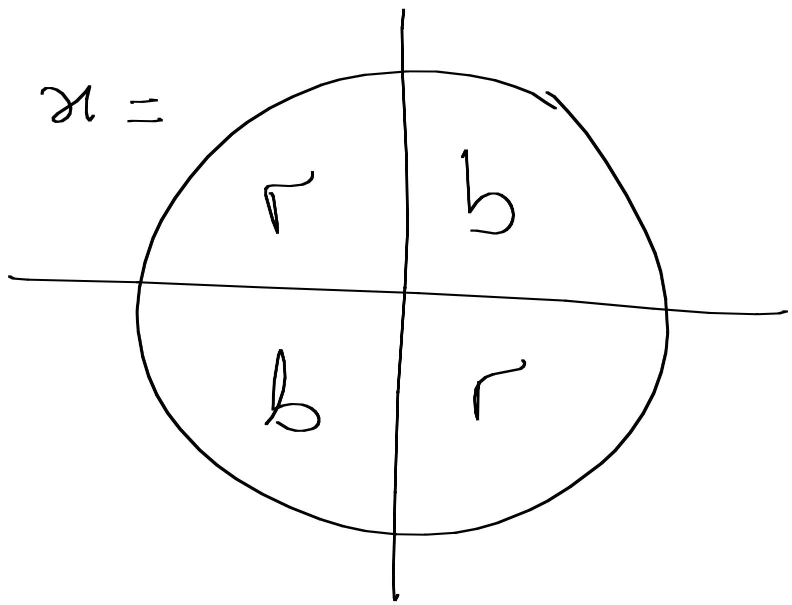
$\underbrace{g_1^{-1} \circ g_2}_{\in S_x} * x = x$

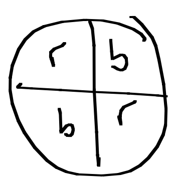
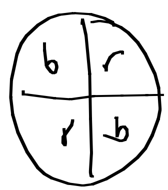
equivalence relation on G



Example 1

$G = [e_0, e_1, e_2, e_3]$

$x =$  $=$

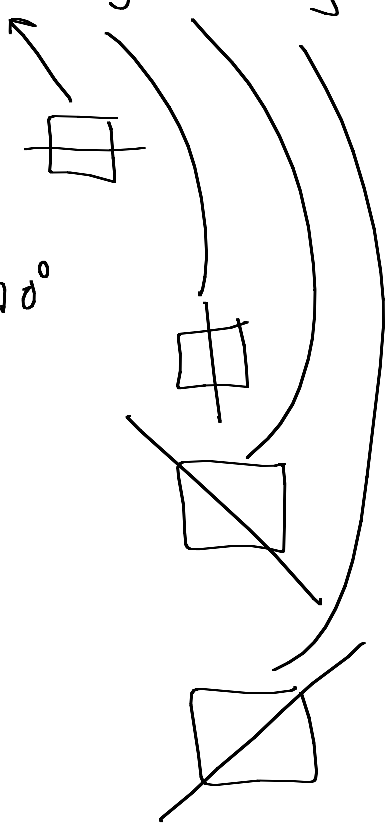
$O_x = \{$  ,  $\}$

$S_x = \{e_0, e_2\}$

$|O_x| = 2$

$G = \{ e, a, b, c, p, q, r, s \}$

$90°$ $180°$ $270°$



x $\begin{array}{|c|c|} \hline r & b \\ \hline b & r \\ \hline \end{array}$

$O_x = \left\{ \begin{array}{|c|c|} \hline r & b \\ \hline b & r \\ \hline \end{array} , \begin{array}{|c|c|} \hline b & r \\ \hline r & b \\ \hline \end{array} \right\}$

$S_x = \{ e, b, r, s \}$

$$\nu_{X,G} = \#\text{ orbits}$$

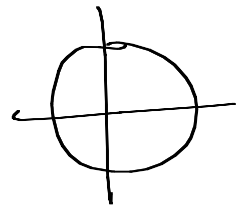
$$= \sum_{x \in X} \frac{1}{|O_x|}$$

$$= \sum_{x \in X} \frac{|S_x|}{|G|}$$

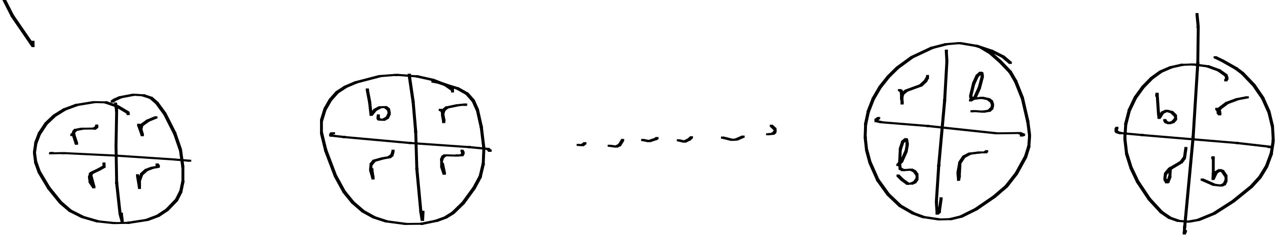
First formula

$$\boxed{\frac{1}{|G|} \sum_{x \in X} |S_x|}$$

<u>Example 1</u>



$$\frac{1}{4}\left(4 + 1 + \cdots + 1 + 2 + 2 + \cdots\right) = 6$$



Example 2

$$\nu_{X,G} = \frac{1}{8}\Big(8 \begin{array}{|c|c|} \hline r & r \\ \hline r & r \\ \hline \end{array} + 2 \begin{array}{|c|c|} \hline b & r \\ \hline r & r \\ \hline \end{array}$$

identity + $\cancel{\square}$

This method is impractical if X is large.

Burnside's Lemma

$$\mathrm{Fix}(g) = \{x : g * x = x\}$$

$g \in G$

$$\nu_{X,G} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

Double Counting argument:

$$\nu_{X,G} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

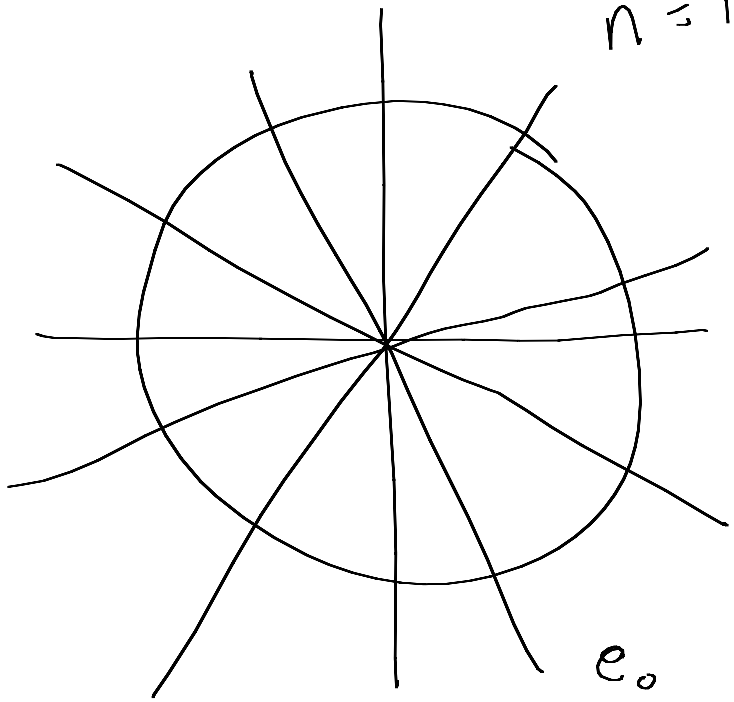
$$= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} A(x,g)$$

$= 1 \quad g * x = x$

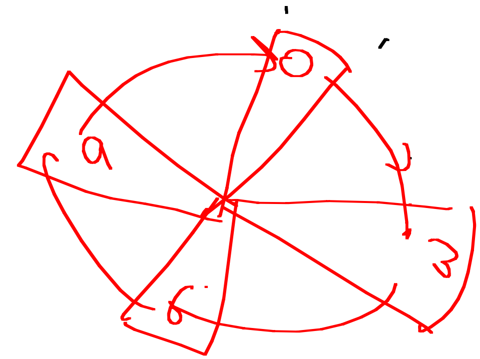
$= 0 \quad g * x \neq x$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\sum_{x \in X} A(x,g)}_{|\mathrm{Fix}(g)|}$$

$n = 12$



$G = \{e_0, e_1, e_2, \dots, e_{11}\}$



$e_0 \quad e_1 \quad + e_2 \quad + e_3 \quad \dots$

$$\nu_{X,G} = \frac{1}{12}\left[2^{12} + 2 + 4 + 8 + \dots\right.$$

$|X| = 2^{12}$