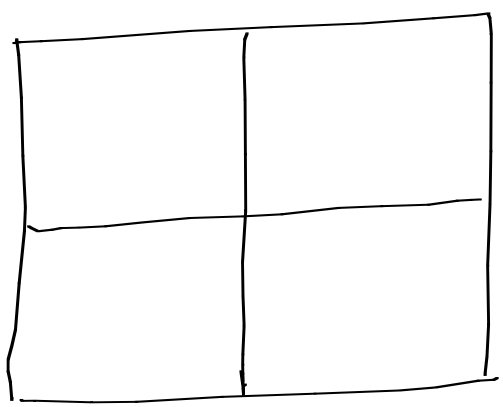
11\17\08

Counting with Symmetries.

2 colors

R & B



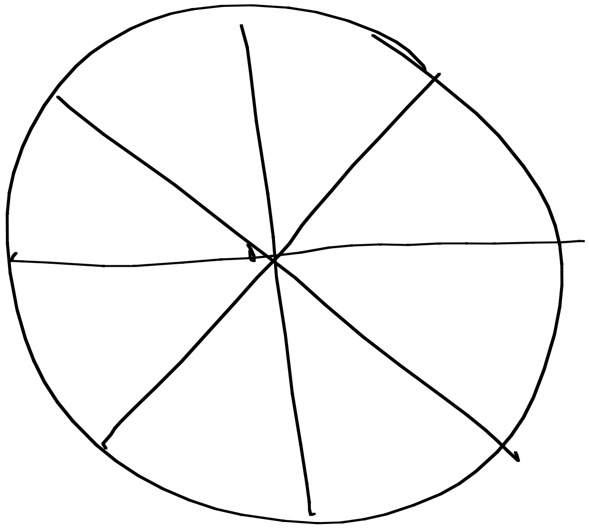
If ↗ is stuck to table: 2^4 colorings

? same coloring?

R	R
B	B

B	R
B	R

Example 1

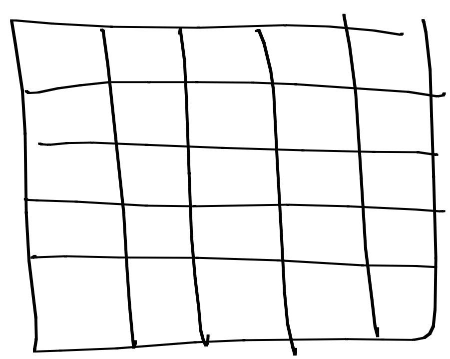


Disc is rotatable.

n sectors

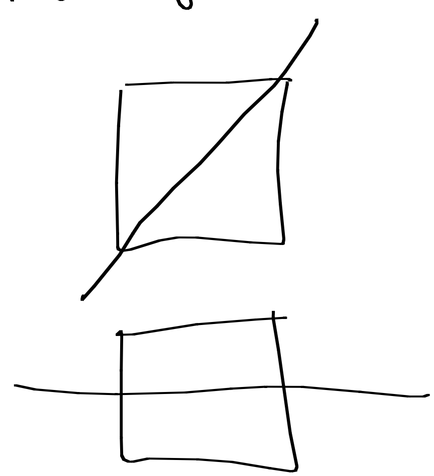
n rotations.

Example 2



(i) Rotate $\leftarrow$ 4 rotations

(ii) Reflect 2



$\downarrow$ 2

General Scenario:

Set X (up to now $X = \{\text{colorings}\}$)

G is a set of permutations of X

G has a group structure:

Given $g_1, g_2 \in G$ we can define a new permutation $g_1 \circ g_2$

$$g_1 \circ g_2 (x) = g_1\left(g_2(x)\right)$$

Properties of G: $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$.

A1: Indentity $\mathbb{1}_X \in G$. $[\mathbb{1}_X(x) = x]$

A2: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

Associativity

A3: $g^{-1} \in G$ for all $g \in G$

GROUP

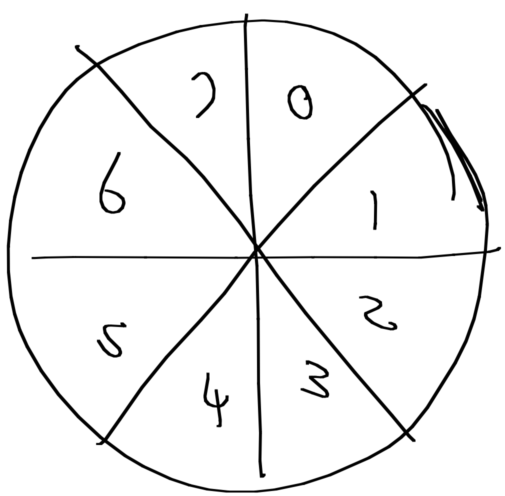
Example 1

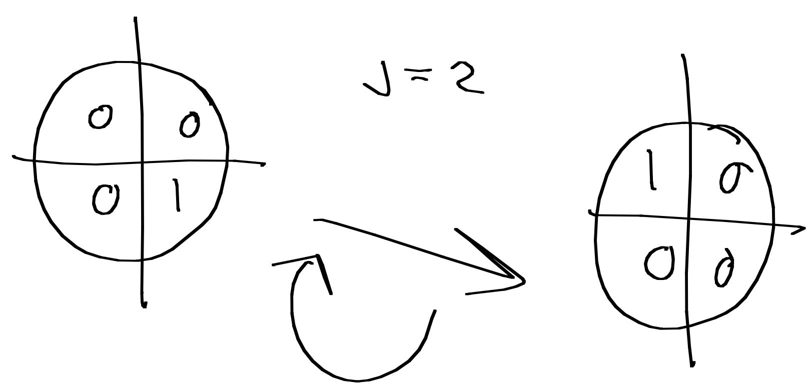
$D = \{0, 1, 2, \ldots, n-1\}$

$X = 2^D$

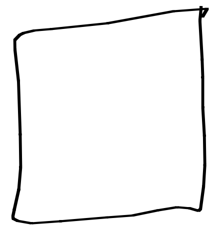
$G = \{e_0, e_1, \ldots, e_{n-1}\}$

e_v rotates by $v \times \frac{2\pi}{n}$





Example 2



$$G = \{e, a, b, c, p, q, r, s\}$$

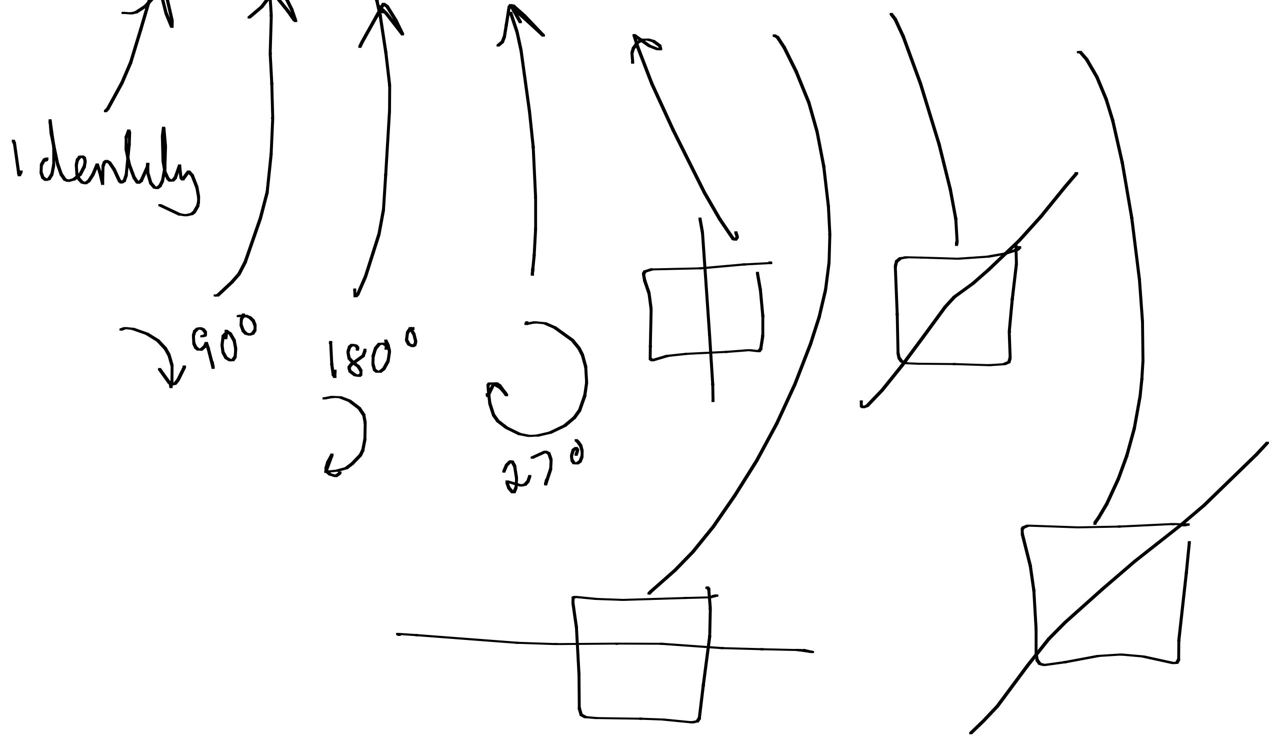
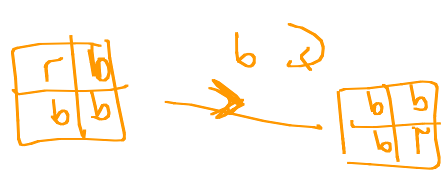


Table:

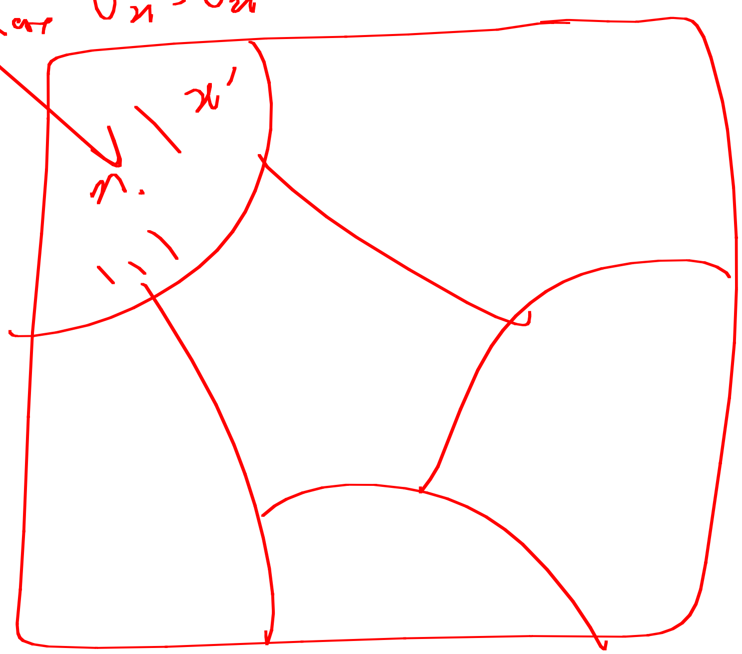


Orbits

$x \in X$, $\quad O_x = \{ y : \exists g \in G,\ y = g * x \}$

Orbits partition X

x, x' have $O_x = O_{x'}$



X

Problem: Count the number of orbits

Orbits partition X

(i) $\bigcup_x O_x = X$

$x \in O_x : 1_X(x) = x$

$g * (h \circ y) = (g \circ h) * y$

Part of our assumptions

(ii) $\underbrace{O_x \cap O_y \neq \emptyset}$ $\Rightarrow O_x = O_y$

$g_1 * x = g_2 * y$ for some g_1, g_2

$\underline{O_x \subseteq O_y}$

$g * x = [g \circ (g_1^{-1} \circ g_2)] * y \in O_y$

Similarly $O_y \subseteq O_x$.