11\14\08

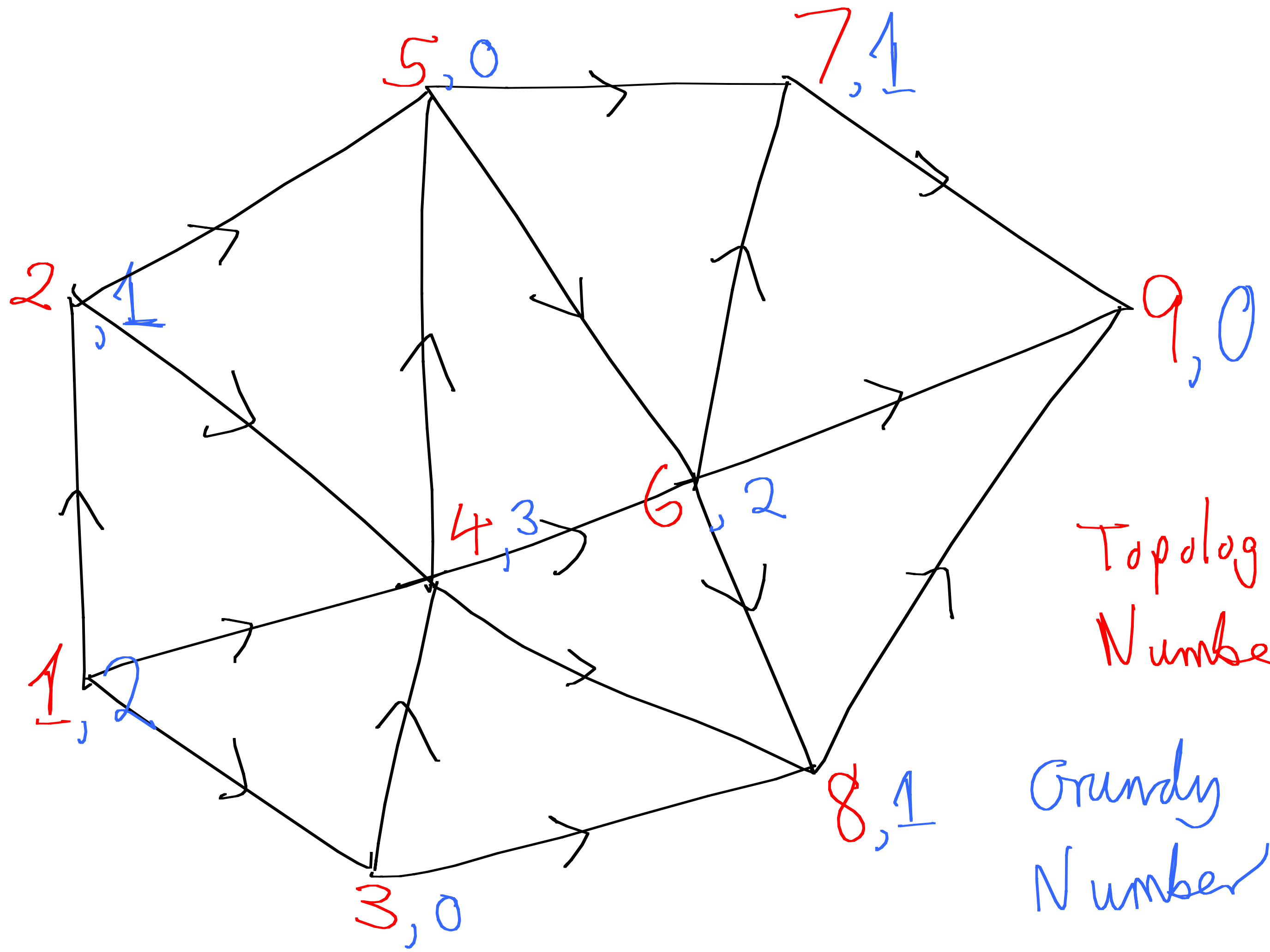
$$S \subseteq \{0, 1, 2, \ldots\}$$

$$\text{mex}(S) = \min\{x \geq 0 : x \notin S\}$$

minimum
excluded

$$\text{mex}(\{0, 1, 3, 8\}) = 2$$

$$\text{mex}(\{17, 46\}) = 0$$



Topological Number

Grundy Number

$g(x) = 0$ iff $x \in P$

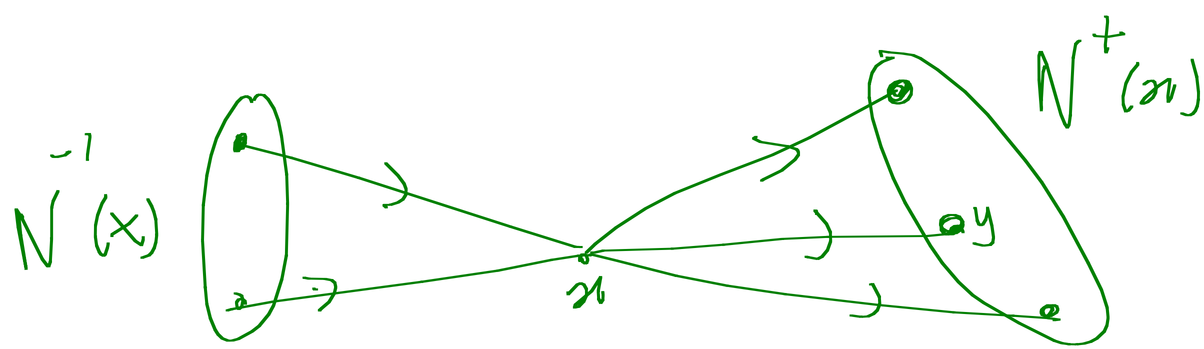
$g(x) = 0$ iff $x \in P$

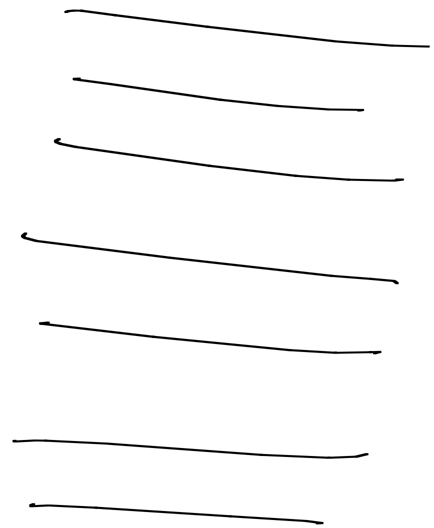
$x \in N$ iff $N^+(x) \cap P \neq \emptyset$

We can show

$g(x) > 0$ iff $\exists\, y \in N^+(x) : g(y) = 0$

$x \in N$ $\quad$ $y \in P$





Moves:

① Any even number, but not whole pile

(ii) Whole pile, if it is odd.

x	0	1	2	3	4	5	6
$g(x)$	0	1	0	2	1	3	2

Claim: $g(2k) = k-1$

$g(2k-1) = k$

<u>Proof</u>: Induction on k.

$$g(2k+2) = \text{mex}\{\underset{k-1}{g(2k)}, \underset{k-2}{g(2k-2)}, \dots, \underset{0}{g(2)}\}$$

$$= k$$

$$g(2k+1) = \text{mex}\{\underset{k}{g(2k-1)}, \underset{k-1}{g(2k-3)}, \dots, \underset{1}{g(1)}, \underset{0}{g(0)}\}$$

$$= k+1.$$

Grundy Addition $\oplus$

"binary addition without carry"

$$7 \oplus 11$$

$$
\begin{array}{rrcccc}
 & 7 = & 0 & 1 & 1 & 1 \\
\oplus & 11 = & 1 & 0 & 1 & 1 \\
\hline
 & 12 & 1 & 1 & 0 & 0
\end{array}
$$

$G_1, G_2, \ldots, G_p$

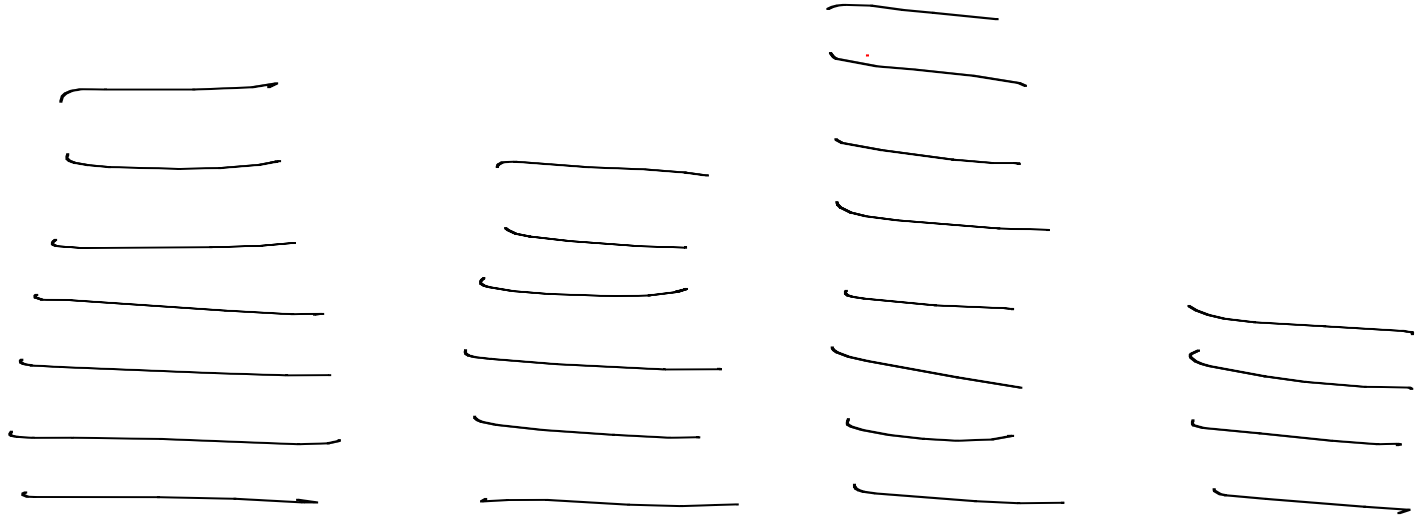
$g_1, g_2, \ldots, g_p$ are SG-numbers

$g(x_1, x_2, \ldots, x_p) =$

$g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$

"On numbers and games" ONAG

by John Conway



7 6 8 4

One pile num: $g(x) = x$

7	=	0	1	1	1
6	=	0	1	1	0
8	=	1	0	0	0
4	=	0	1	0	0
$\oplus$		1	1	0	1

Winning Position

$1000 \rightarrow 0101$ i.e. remove 3

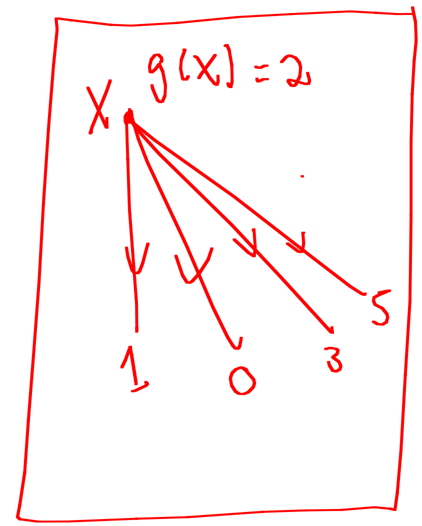
winning move.

Only need to prove this for $p = 2$ and then use induction.

$$\underbrace{(G_1 \oplus G_2 \oplus G_3)} \oplus G_4$$

Induction

$$(g_1 \oplus g_2 \oplus g_3) \oplus g_4$$



To prove: Given g_1, g_2. Show $g_1 \oplus g_2$ is the SG function

A1: $x \in X$ and $g(x) = b > a \Rightarrow \exists\, x' \in N^+(x)$ with $g(x') = a$.

A2: $x \in X$ and $g(x) = b \Rightarrow \nexists\, x'$ . . . . .

$g(x) = b$, want to see a

$$d = a \oplus b$$

$$\begin{aligned} a &= d \oplus b \\ &= d \oplus g_1(x_1) \oplus g_2(x_2) \end{aligned}$$

We show

$$d \oplus g_1(x_1) < g_1(x_1)$$

$\exists x_1' \in N_1^+(x_1)$ s.t. $g_1(x_1') = d \oplus g_1(x_1)$

$\Downarrow$

$g(x_1', x_2) = a$

or

$$d \oplus g_2(x_2) < g_2(x_2)$$

Suppose $2^{k-1} \leq d < 2^k$

$\ldots 0 \; 0 \; 1 \; x \; x \; x \ldots x \qquad d$

$\uparrow$

position k

$d_k = a_k \oplus b_k \qquad a_k = 0 \;\&\; b_k = 1$

$a < b \qquad g_1(x_1)$ has $\underline{1}$ in pos. k

or

$g_2(x_2)$ has $\underline{1}$ in pos k

$d \oplus g_1(x_1) < g_1(x_1)$