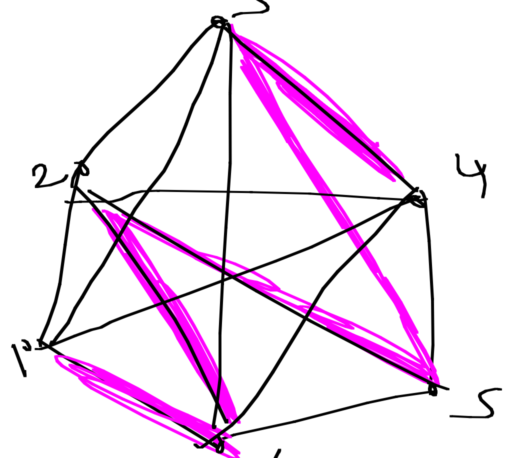
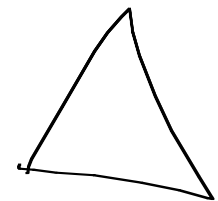
10\29\08

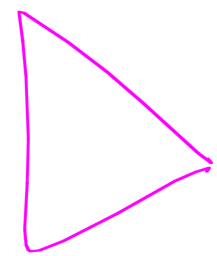
Ramsey Theory



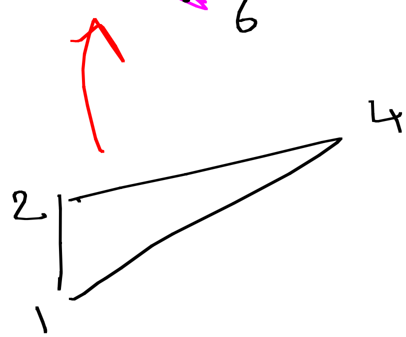
Either $\exists$

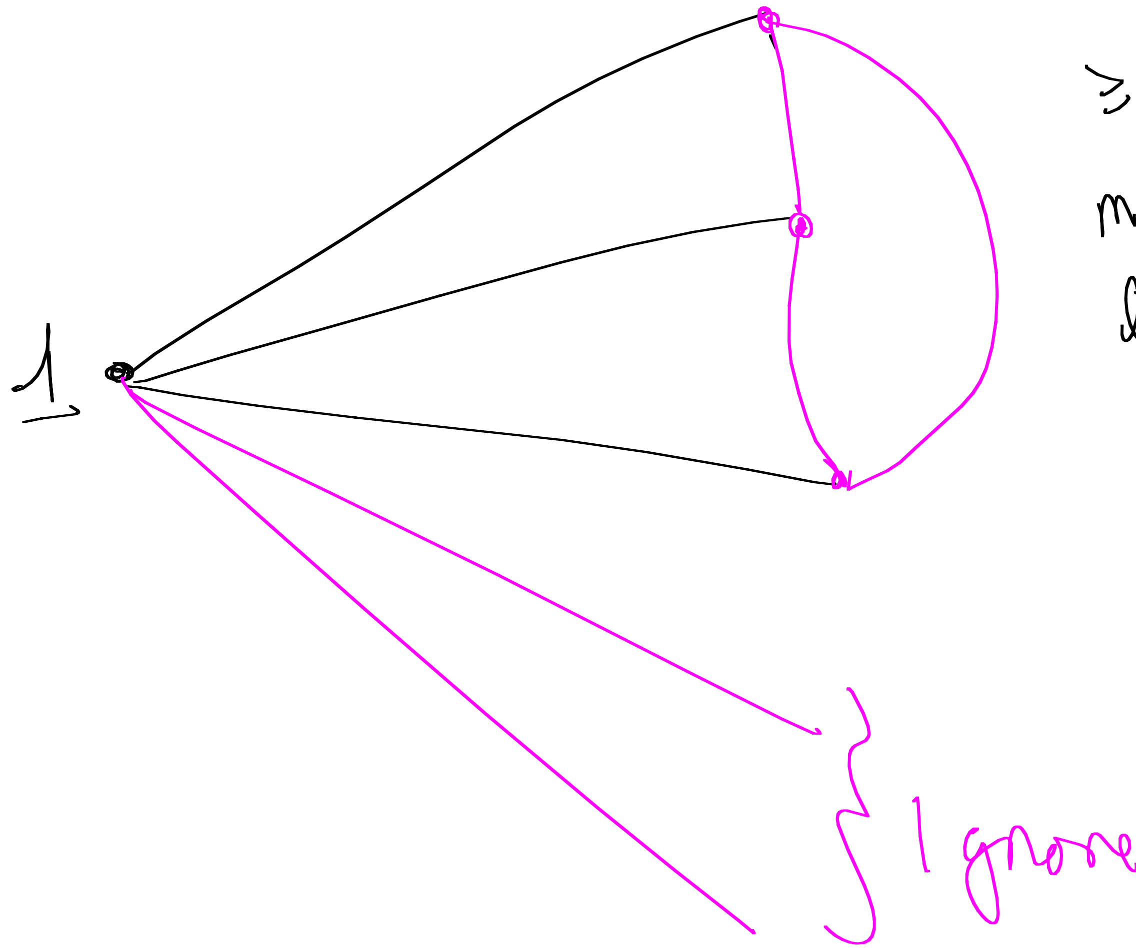


or

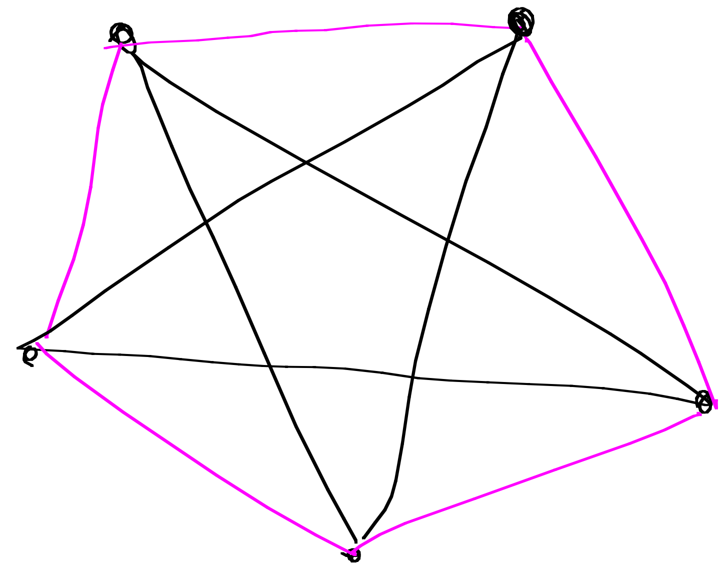


or both





In fact there are ≥ 2 mono-chromatic triangles



No mono-chromatic $\triangle$

$$R(3,3) = 6$$

Ramsey's Theorem:

A clique is a complete subgraph

For positive integers k, l

$\exists$ an integer $R(k, l)$ such that

if $N \geq R(k, l)$

and we color the edges of K_N

Red and Blue, then either there is

a Red k-clique or

a Blue l-clique or both

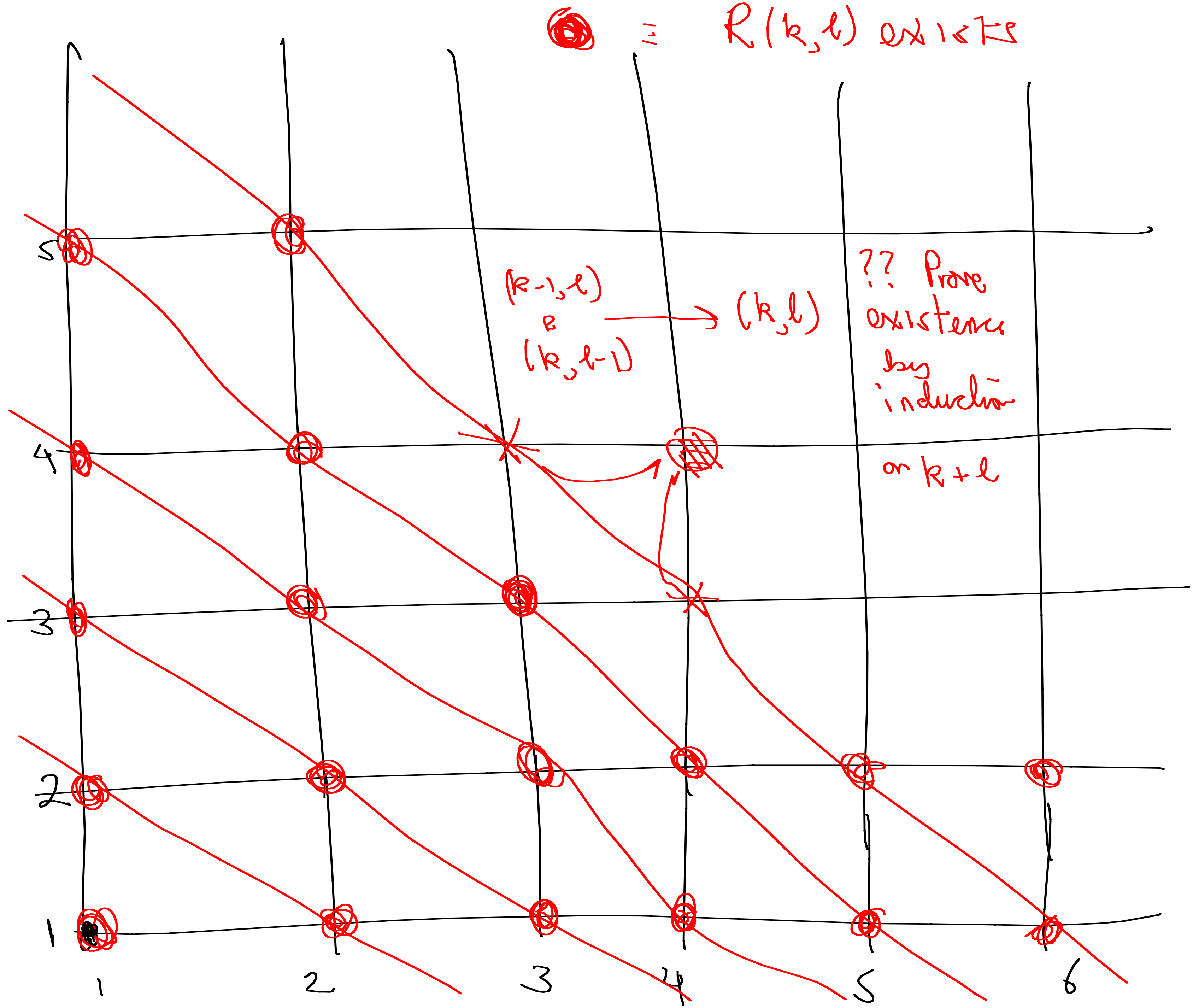
$$R(1,k) = 1 = R(k,1)$$

K_1

$$R(\overset{\text{Red}}{2}, \overset{\text{Blue}}{k}) = k$$

$K_{k-1} \leftarrow$ color all edge Blue –
No Red 2-clique = edge
No Blue k-clique

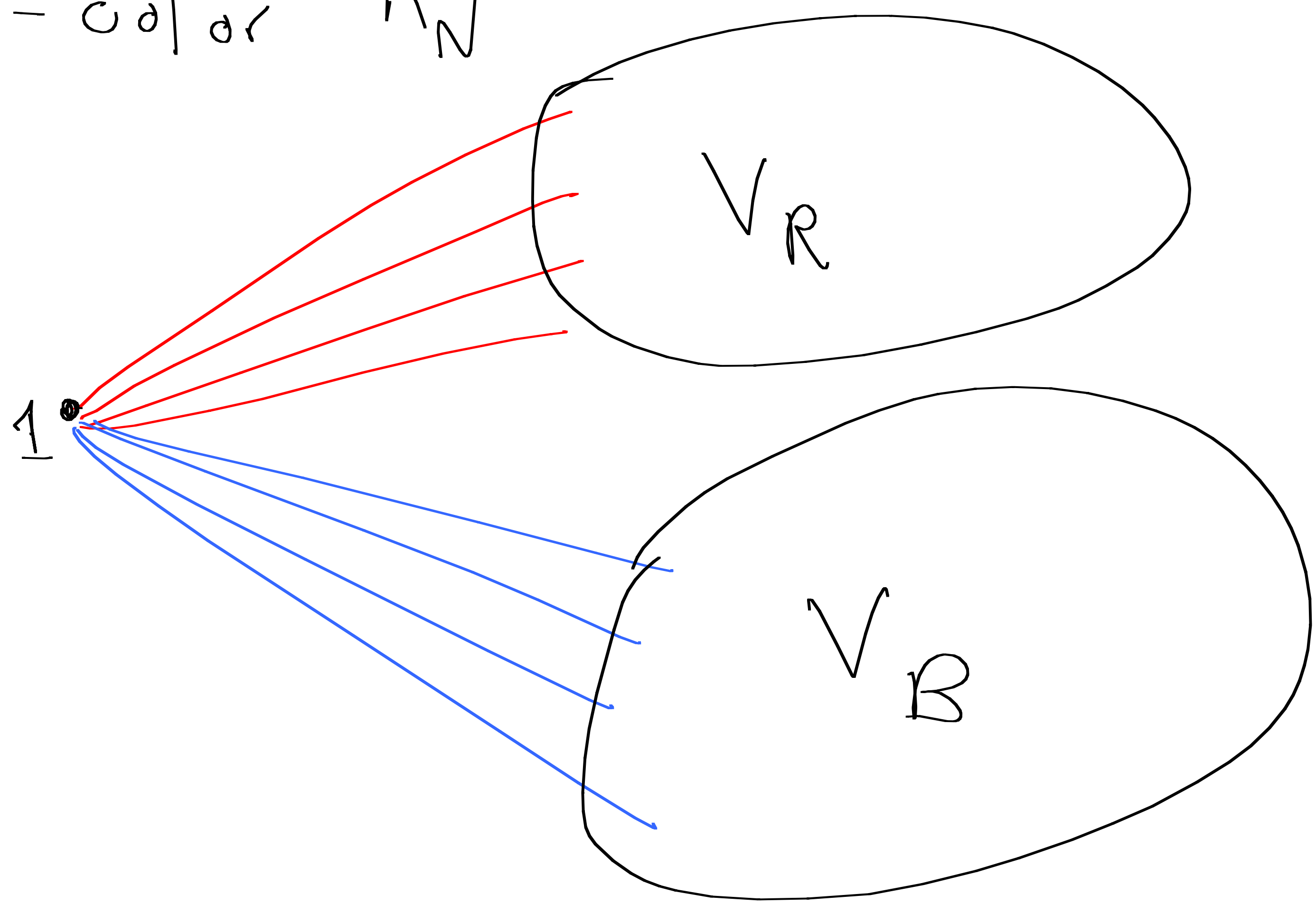
$K_k \leftarrow$
(i) I use Red $\rightarrow \exists$ Red 2-clique
(ii) All Blue $\longrightarrow \exists$ Blue k-clique


● = $R(k,\ell)$ exists
$(k-1,\ell)$ & $(k,\ell-1)$ → (k,ℓ)
?? Prove existence by induction on $k+\ell$
1
2
3
4
5
6
1
2
3
4
5

Erdös & Szekeres

$$R(k,\ell) \le \underbrace{R(k,\ell-1)+R(k-1,\ell)}_{N}$$

2-color K_N



$$|V_R| + |V_B| = N - 1$$

$$= R(k, l-1) + R(k-1, l) - 1$$

Either $|V_R| \geq R(k-1, l)$

or $|V_B| \geq R(k, l-1)$

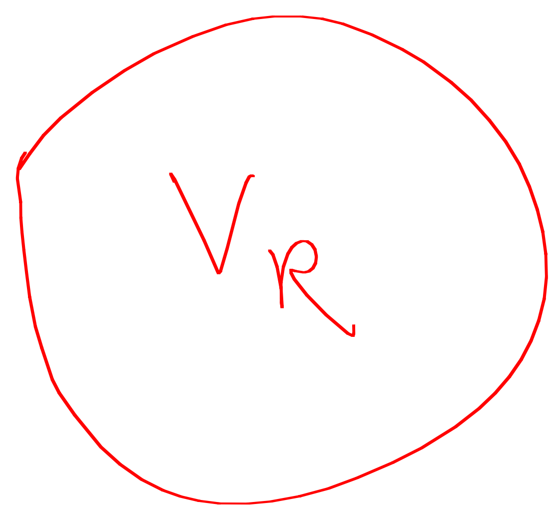
If not then

$$|V_R| \leq R(k-1, l) - 1$$
$$|V_B| \leq R(k, l-1) - 1$$
$$\Rightarrow |V_R| + |V_B| \leq N - 2$$

Either $|V_R| \geq R(k-1, \ell)$ (a)

or $|V_B| \geq R(k, \ell-1)$ (b)

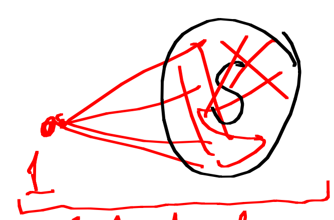
Case (a) :



The edges of V_R have been 2-colored,

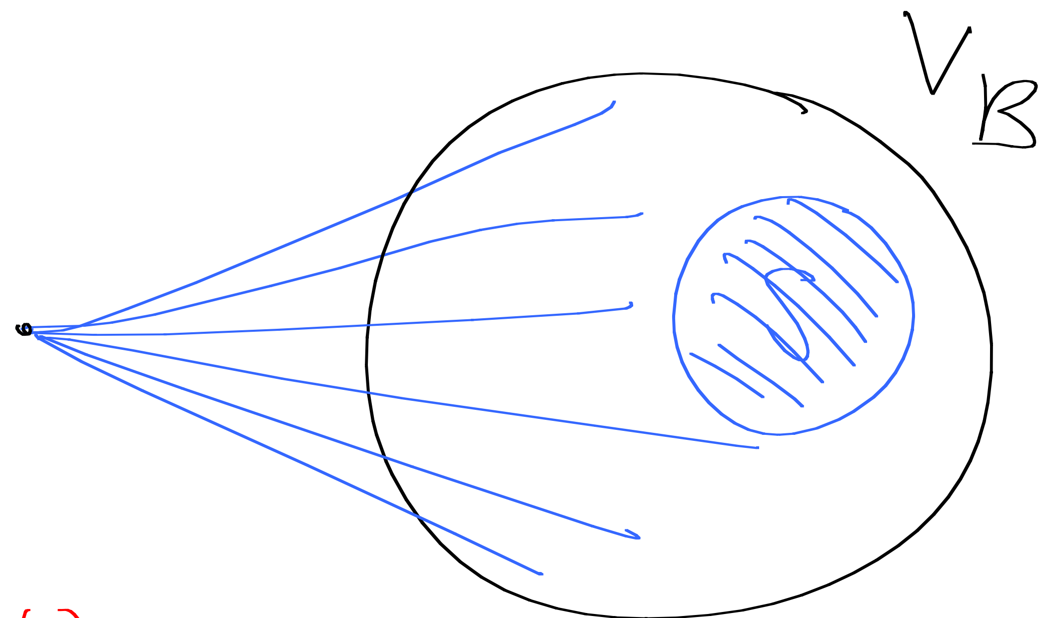
Either (i) $\exists$ Blue ℓ-clique ✓

(ii) $\exists$ Red $(k-1)$-clique S



<u>Case 1(b)</u>

$$|V_B| \geq R(k, l-1)$$



Either (i) $\exists$ Red k-clique ✓

(ii) $\exists$ Blue $(l-1)$-clique S

$S \cup \{1\}$ is a Blue l-clique.

$R(3,3) = 6$

$R(4,4) = 18$

$R(5,5) = ??$

$R(k,k)$, $k \geq 5$ in unknown.

Hard to compute

Upper bound on $R(k,l)$

$$R(k,l) \leq \binom{k+l-2}{k-1} \quad (*)$$

$$R(k,k) \leq \binom{2k-2}{k-1} < 4^k$$

Later we see that

$$2^{k/2} < R(k,k) < 4^k$$

Proof of upper bound:

Induction on $k+\ell$.

True for $k+\ell \leq 5$ — just check.

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell)$$

$$\underset{\text{induction}}{\leq} \binom{k+\ell-3}{k-1} + \binom{k+\ell-3}{k-2}$$

$$= \binom{k+\ell-2}{k-1}.$$

$\square$