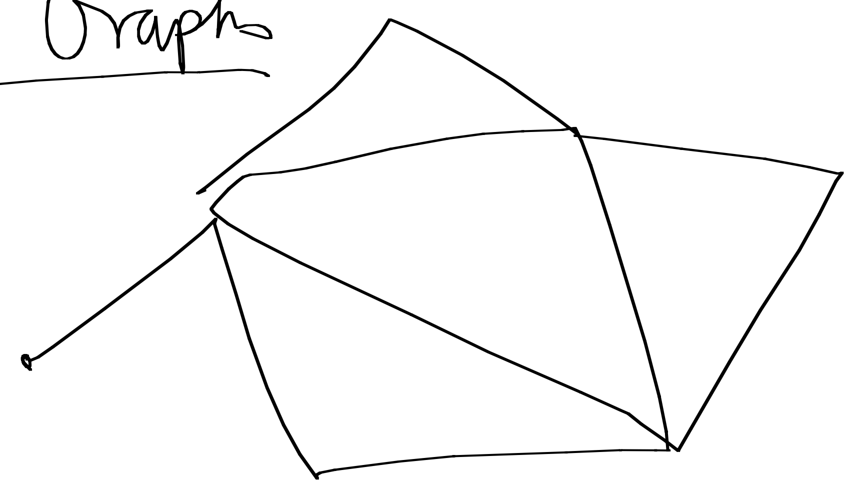
10/20/08

# Graph Theory

## Eulerian Graphs



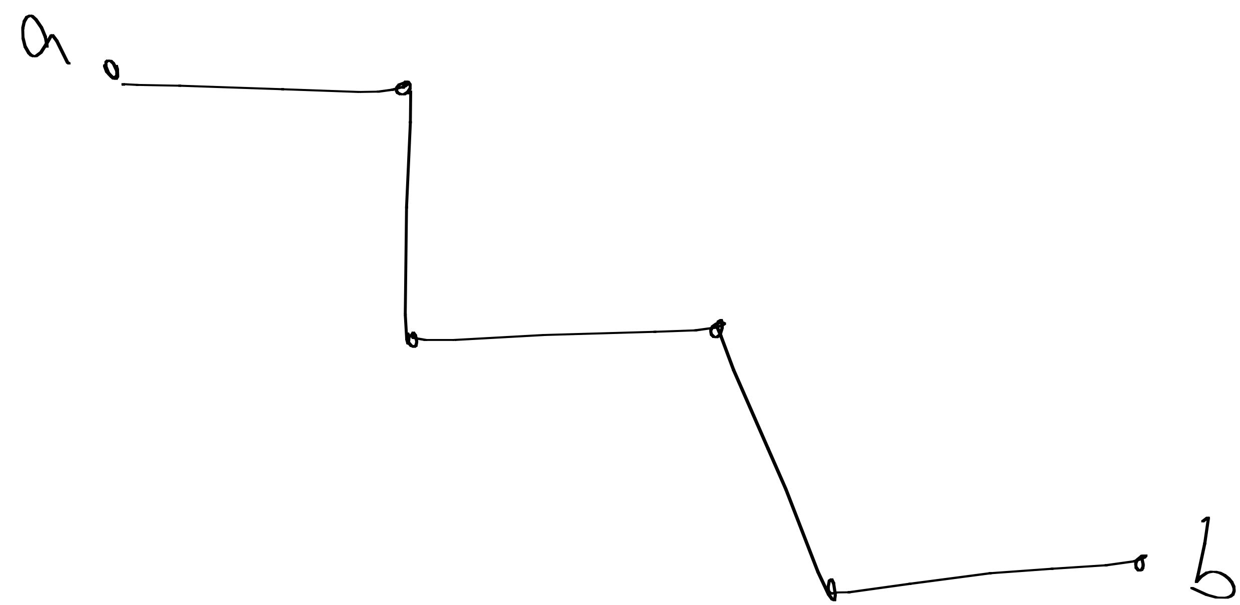
$$\sum_v d_G(v) = 2|E|$$

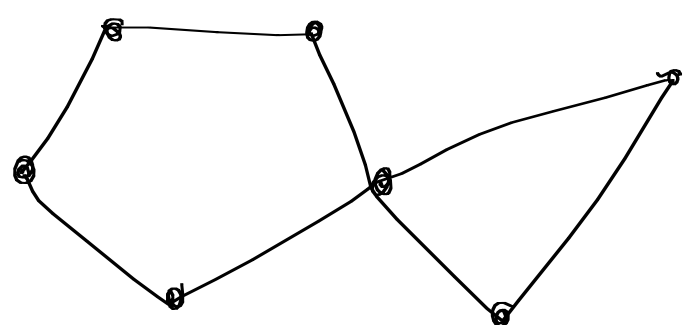
$$\begin{array}{cc}
 & \begin{array}{cccccc} e_1 & e_2 & \cdots & e_i & \cdots & e_n \end{array} \\
\begin{array}{c} \\ v \\ \\ \end{array} & \begin{bmatrix} & & & 1 & & \\ & 1 & & 1 & & 1 \\ & & & & & \end{bmatrix}
\end{array}$$

# of 1's $= \sum_{\text{rows}} = \sum_v d_G(v)$

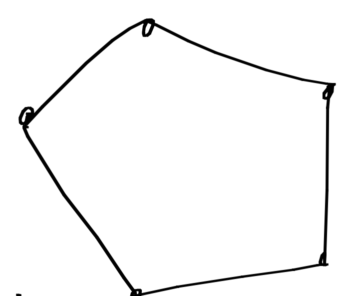
$= \sum_{\text{column}} = 2|E|$

# Paths, Walks



CLOSED WALK



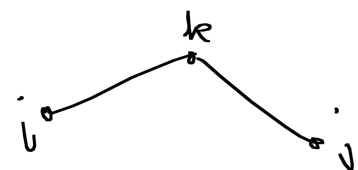
CYCLE

$A$ is the adjacency matrix of graph $G$.

$A_{ij}$ = # edges joining $i$ & $j$ [0 or 1]

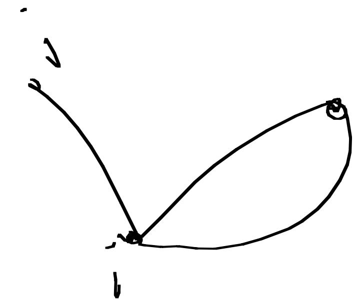
$$B = A^2$$

$$B_{ij} = \sum_k \underbrace{A_{ik} A_{kj}}_{1 \text{ iff}}$$



= # ~~paths~~ walks of length 2 between $i$ and $j$

$(A^3)_{ij}$




Claim: $(A^k)_{ij}$ = # walks of length $k$ from $i$ to $j$

<u>Proof</u>

Induction on $k$.

$k=1$ — easy

## Inductive Step

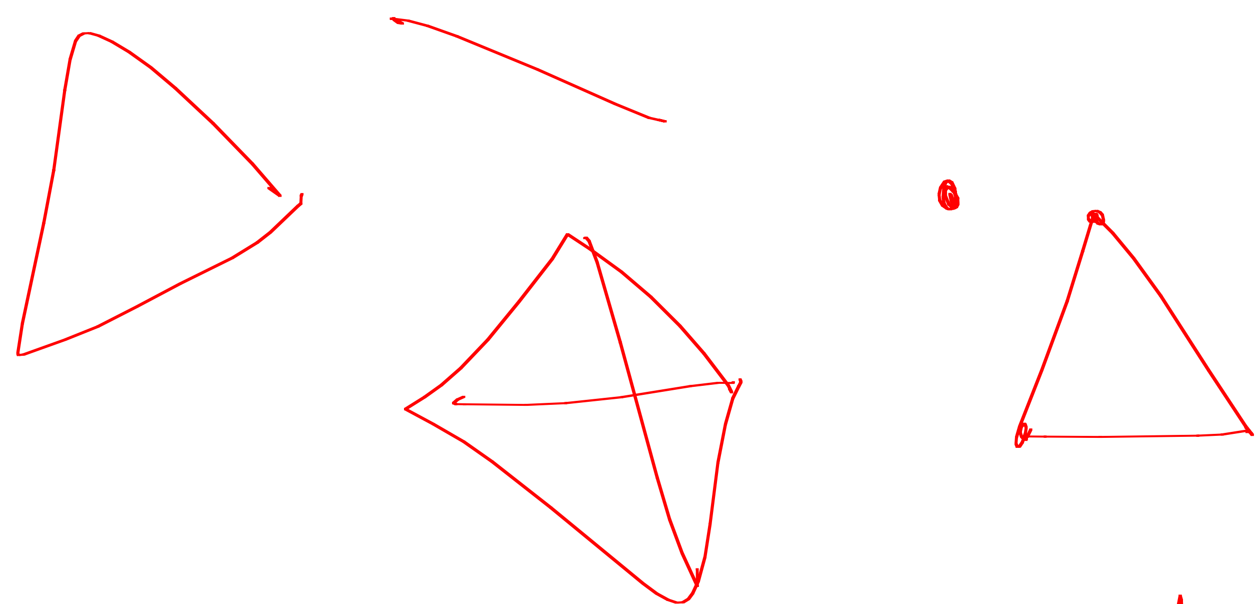
$$A^{k+1} = A^k A$$

$$\left(A^{k+1}\right)_{ij} = \sum_{\ell} \left(A^k\right)_{i\ell} A_{\ell j}$$

# paths of length k from i to $\ell$

$\therefore$ Walks of length $k+1$ from $i$ to $j$

<u>Graph</u>



5 connected components

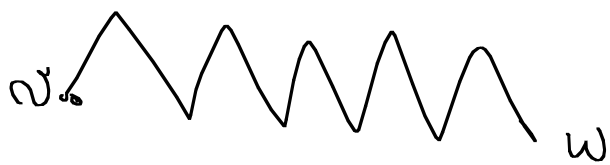
$i \sim j$ if $\exists$ walk from $i$ to $j$

Equivalence relation

$v \sim v$ — Reflexive

$v \sim w$ — Symmetric

$\Rightarrow w \sim v$




$v \sim w,\ w \sim x$

$\Rightarrow v \sim x$ — transitive

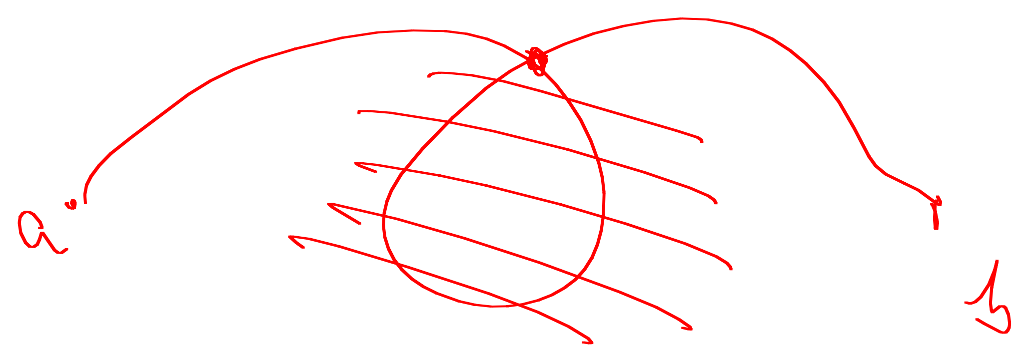
Equivalence classes of $\sim$ are called **components**.

$u, v \in$ same component $\Longleftrightarrow \exists$ walk from $u$ to $v$.

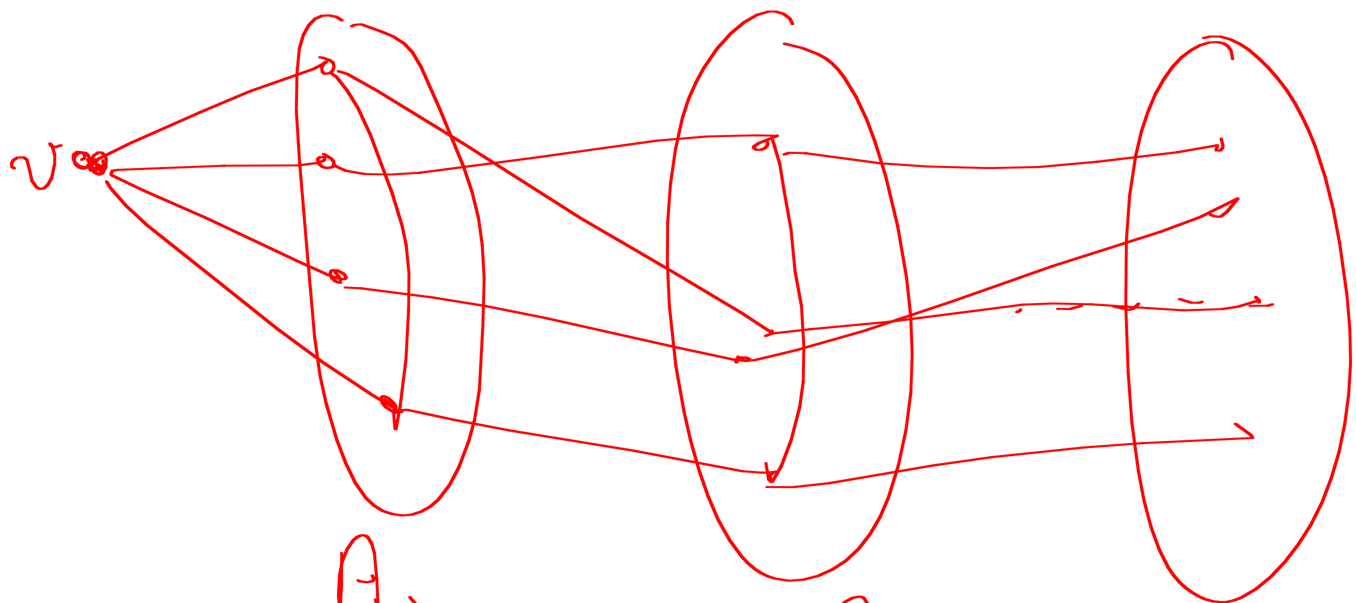
If $G$ has one equivalence class then $G$ is **connected**

If $\exists$ a walk from $a$ to $b$, then $\exists$ a path from $a$ to $b$.

Let $P$ be a walk with fewest edges. — must be a path.



# Breadth First Search




$A_1 = \{$ neighbors of $v \}$

$A_2 = \{$ vertices at distance 2 from $v \}$

$A_3$

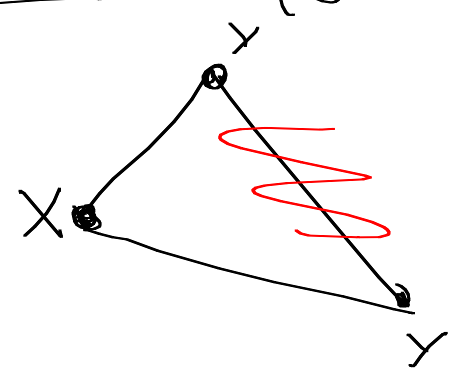
**Thm**

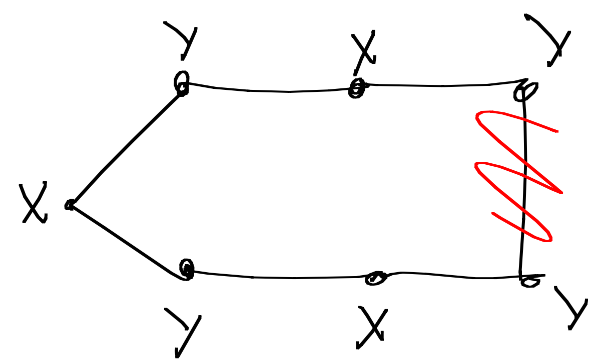
G is bipartite iff G has no odd cycles.

**Proof**

(a) Bipartite $\Rightarrow$ no odd cycles

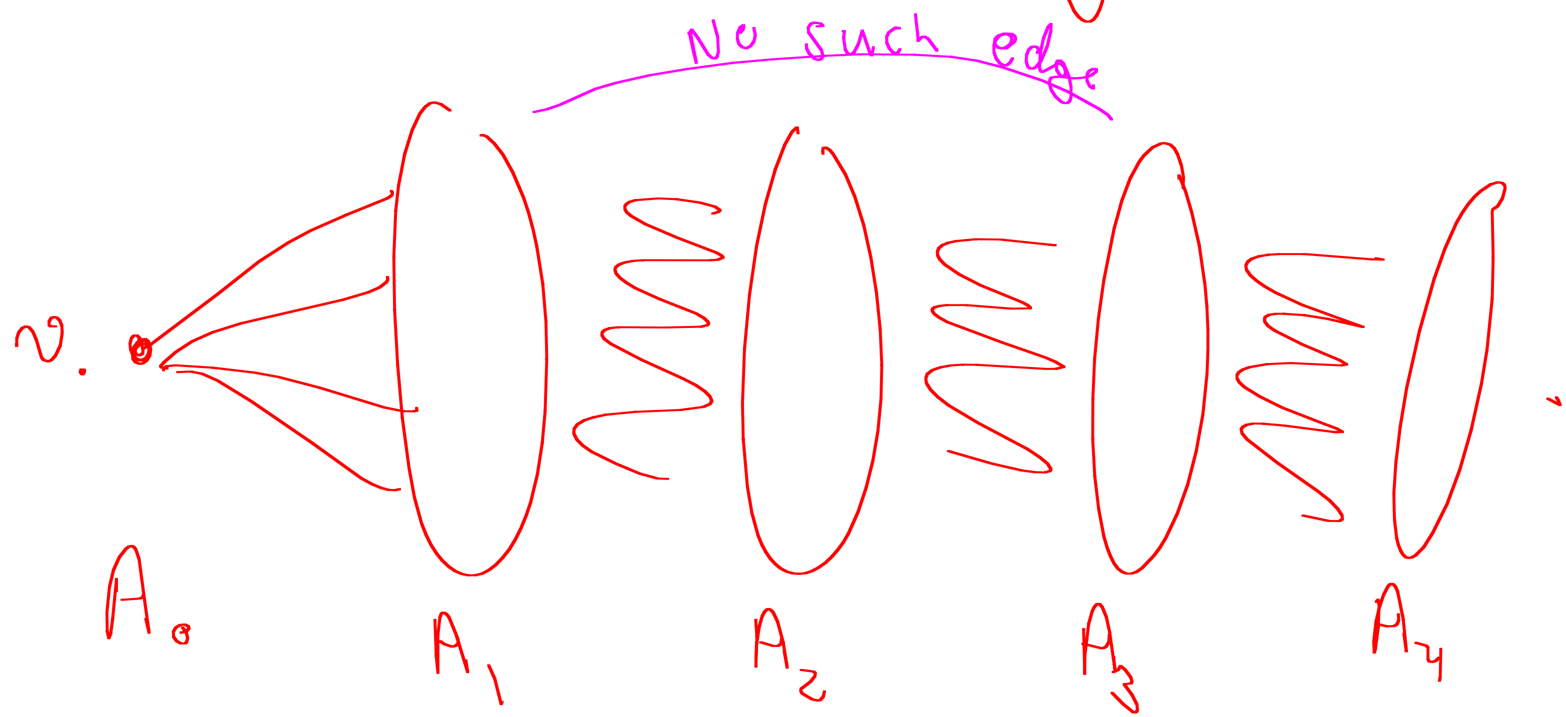
$G = (X \cup Y, E)$




NO ODD CYCLES

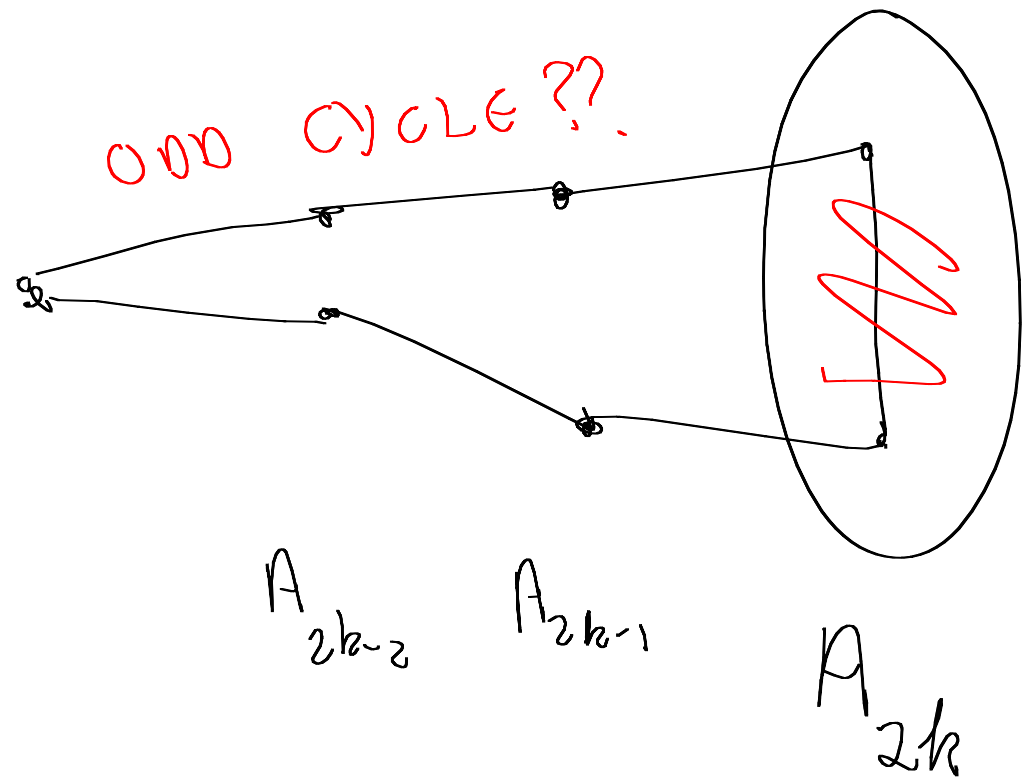
(b) G has no odd cycles.




Claim: $X = A_0 \cup A_2 \cup A_4 \cup \ldots$

$Y = A_1 \cup A_3 \cup A_5 \cup \ldots$

Bipartition


ODD CYCLE??
$A_{2k-2}$
$A_{2k-1}$
$A_{2k}$