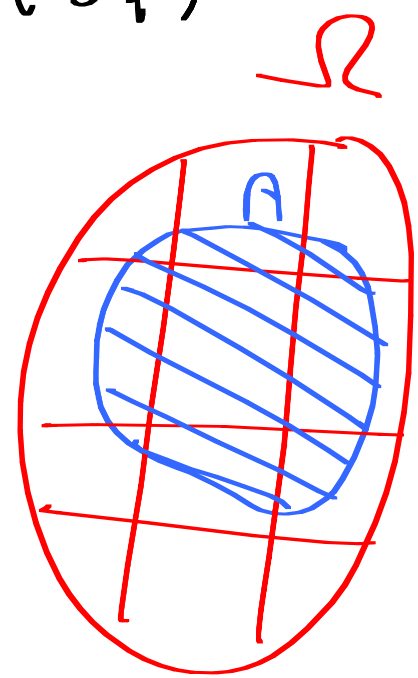
10\1\08

$B_1, B_2, \ldots B_n$ is a partition of Ω.

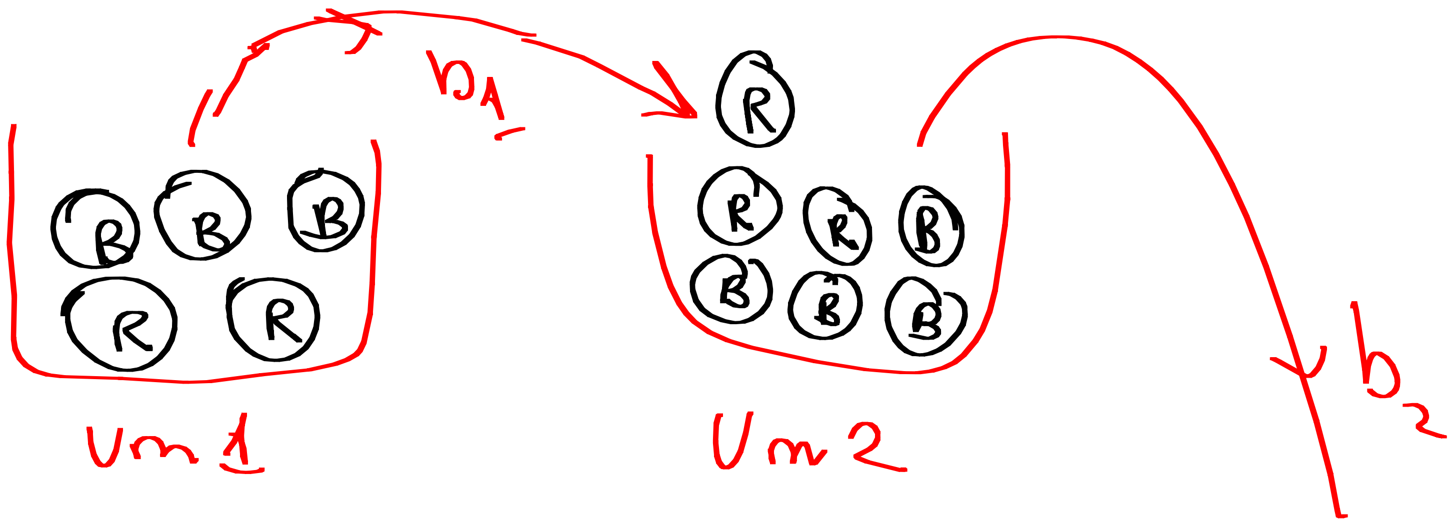
A is an event.

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$$



$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$= \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$$



$Pr(b_2$ is Blue ?)

$A = \{b_2 \text{ is Blue}\}$

$B_1 = \{b_1 \text{ is Blue}\}$ $B_2 = \{b_1 \text{ is Red}\}$

$P(B_1) = 3/5$

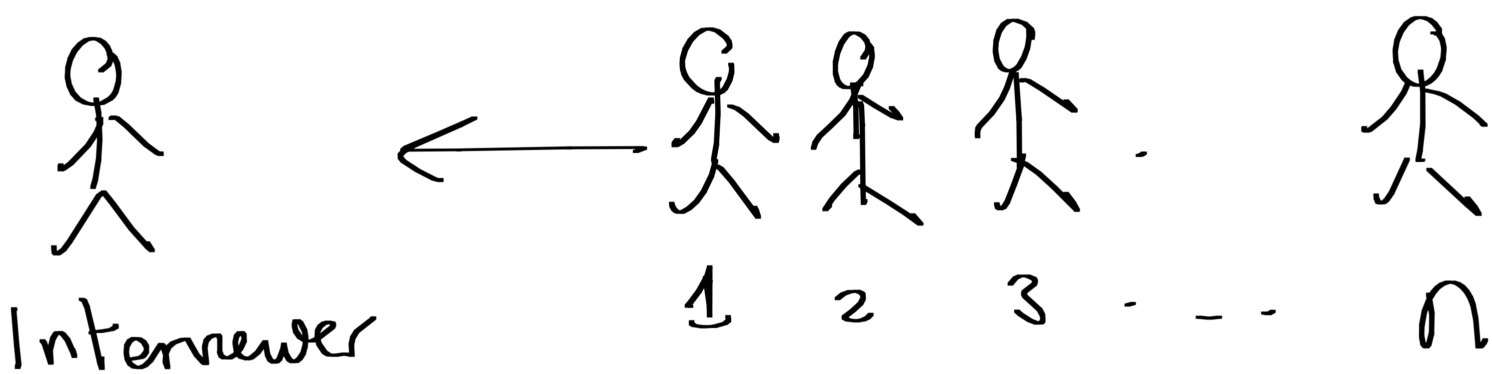
$P(B_2) = 2/5$

$P(A|B_1) = 5/8$

$P(A|B_2) = 1/2$

$P(A) = \frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{1}{2} = 23/40.$

Secretary Problem



Interviewees come in one at a time.
If job is offered they accept.
If not offered they cannot be hired later

Assume that interviewer can evaluate precisely and that interviewees come in a random order.

Strategy: Choose $m \leq n$

Interview $1, 2, \ldots, m$ without making hire.

For $i = m+1, m+2, \ldots, n$ hire person i if they are better than $1, 2, \ldots, i-1$.

?? What is probability of hiring best person ??

$S = \{ P_a \text{ t choose best applicant} \}$

$P_i = \{ i \text{ th person is best} \}$

$$P(S) = \sum_{i=1}^{n} P(S \mid P_i)\, \Pr(P_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} P(S \mid P_i)$$

$$= \frac{1}{n} \sum_{i=m+1}^{n} P(S \mid P_i)$$

Suppose $i > m$.

What is $P(S \mid P_i)$

= Probability Pat has not chosen any of $m+1, \dots, i-1$

= Probability the best of $1, 2, \dots, i-1$ is $\leq m$

$$= \frac{m}{i-1}$$

$$\Pr(S) = \frac{1}{n} \sum_{i=m+1}^{n} \frac{m}{i-1}$$

$$\sum_{j=1}^{k} \frac{1}{j} \sim \log k$$

$$= \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$$

What is best choice for m?

Take $m = \alpha n$

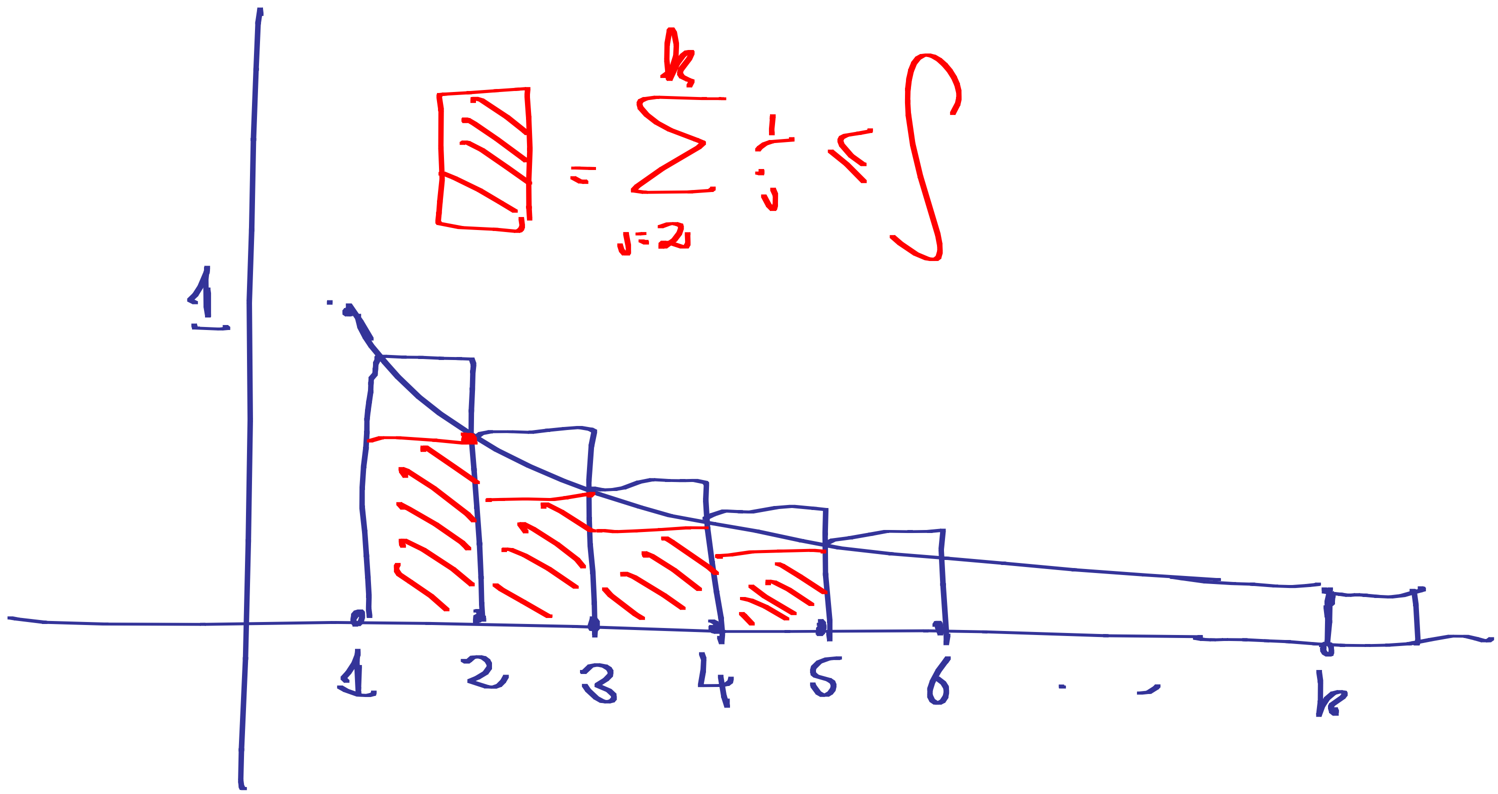
$$\Pr(S) = \alpha \sum_{i=\alpha n+1}^{n} \frac{1}{i-1}$$

$$\sim \alpha \left[\log_e n - \log_e \alpha n \right]$$

Aside

$$\sum_{j=1}^{k} \frac{1}{j} \geq \int_1^{k+1} \frac{dx}{x} = \log k+1$$

$$\boxed{} = \sum_{j=2}^{k} \frac{1}{j} \leq \int$$



$$\log k+1 - \log k = \log 1 + \frac{1}{k} \leq \frac{1}{k}$$

$$P(S) \sim \alpha \log_e \frac{n}{\alpha n}$$

$$= \alpha \log 1/\alpha \leftarrow \underline{f(\alpha)}$$

Maximise $f(\alpha)$: $\quad = -\alpha \log \alpha$

$$f'(\alpha) = -\log \alpha - \alpha \cdot \frac{1}{\alpha} = 0$$

$$\log_\alpha \alpha = -1$$

$$\alpha = 1/e$$

$$P(S) = 1/e$$

Choose two sets S & T uniformly from $\{1, 2, \ldots, n\}$

$$\Pr(|S| = |T| \ \& \ \underbrace{S \cap T = \emptyset}_{A})$$

$B_X = \{S = X\}$ for $X \subseteq [n]$

$$P(A) = \sum_X \Pr(A \mid B_X)\, P(B_X)$$

$\binom{n-|X|}{|X|} \times \frac{1}{2^n}$ $\qquad$ $\frac{1}{2^n}$