10\12\08

Markov Inequality

X is a random variable, $X \in \{0, 1, 2, \dots\}$

$E(X) < \infty$

For any $t \geq 1$

$$\Pr(X \geq t) \leq \frac{E(X)}{t}$$

$$E(X) = \overbrace{E(X \mid X < t)\, Pr(X < t)}^{\geq 0}$$

$$+ \; E(X \mid X \geq t)\, Pr(X \geq t)$$

$$\geq \; E(X \mid X \geq t)\, Pr(X \geq t)$$

$$\geq \; t \;\; Pr(X \geq t)$$

Example: If $t = 1$

$$Pr(X \geq 1) \leq E(X)$$

First Moment Method

m balls (distinguishable)

$\to n$ boxes

$Z = \#$ of empty boxes

$m \geq (1+\epsilon)\, n \log_e n$

$$E(Z) = n\left(1 - \frac{1}{n}\right)^m \qquad \boxed{1+x \leq e^x}$$

$$\leq n e^{-m/n}$$

$$\leq n e^{-(1+\epsilon)\log n}$$

$$= n^{1-1-\epsilon} = n^{-\epsilon} \to 0$$

$$\Pr(Z \geq 1) \leq n^{-\epsilon}$$

$\exists$ empty box

Variance

$E(Z) = \mu$

$$\mathrm{Var}(Z) = E(|Z-\mu|^2)$$

$$= \vdots$$

$$= E(Z^2) - \mu^2$$

$$Z = B_{n,p} \qquad \mu = np$$

$$\mathrm{Var}(B_{n,p}) = \underbrace{\sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}}_{E(Z^2)} - (np)^2$$

$k(k-1) + k$

$$= \sum_{k=\cancel{1}\,2}^{n} \underbrace{k(k-1)\binom{n}{k}}_{\frac{n!}{(k-2)!\,(n-k)!} = n(n-1)\binom{n-2}{k-2}} p^k (1-p)^{n-k} + np - (np)^2$$

$$= n(n-1)p^2 \underbrace{\sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k}}_{(1-p+p)^{n-2}} + np - (np)^2$$

$$= n(n-1)p^2 + np - n^2p^2 = \boxed{np(1-p)}$$

Chebyshev Inequality

$$\sigma \doteq \sqrt{\text{Variance}}$$

standard deviation

$$P(|Z-\mu| \geq t\sigma) \leq \frac{1}{t^2}$$

$$P((Z-\mu)^2 \geq t^2\sigma^2)$$

$$\leq \frac{E((Z-\mu)^2)}{t^2\sigma^2} = \frac{\sigma^2}{t^2\sigma^2} = \frac{1}{t^2}$$

Markov Inequality

$$\Pr\left[\,|B_{n,p} - np| \geq \epsilon np\,\right]$$

$$\leq \frac{\mathrm{Var}(B_{n,p})}{\epsilon^2 n^2 p^2}$$

$$= \frac{np(1-p)}{\epsilon^2 n^2 p^2} = \frac{1-p}{\epsilon^2 n p}$$

Hoeffding Inequality

(Simple Case: Chernoff bounds)

$X_1, X_2, \ldots, X_n$ are <u>independent</u>

$$\Pr(X_i = 1) = \frac{1}{2}$$

$$\Pr(X_i = -1) = \frac{1}{2}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$E(X) = 0$$

$$\Pr(|X| \geq t) \leq 2e^{-t^2/2n}$$

Fix $\lambda > 0$ (choose it later)

$$\Pr(X \geq t) = \Pr(e^{\lambda X} \geq e^{\lambda t})$$

Markov Inequality

$$\leq e^{-\lambda t} E(e^{\lambda X})$$

$$E(e^{\lambda X}) = E\left(\prod_{i=1}^{n} e^{\lambda X_i}\right)$$

X, Y indep.

$E(XY) = E(X)E(Y)$

Independence

$$= \prod_{i=1}^{n} E(e^{\lambda X_i})$$

$$E(e^{\lambda X_i}) = \frac{e^{-\lambda} + e^{\lambda}}{2}$$

$$= 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots$$

$$< e^{\lambda^2/2}$$

$$\Pr(X \geq t) \leq e^{-\lambda t} \cdot \left(e^{\lambda^2/2}\right)^n$$

$$\boxed{-t + \lambda n = 0} \qquad = e^{\underbrace{-\lambda t + \lambda^2 n/2}_{\text{choose } \lambda \text{ to minimize this.}}}$$

$$\Pr(X \geq t) \leq e^{-\lambda t} \cdot \left(e^{\lambda^2/2}\right)^n$$

$$\boxed{-t + \lambda n = 0} \qquad = e^{\underbrace{-\lambda t + \lambda^2 n/2}_{\text{choose } \lambda \text{ to minimise this.}}}$$

Put $\lambda = t/n$

$$\Pr(X \geq t) \leq e^{-t^2/2n}$$