21-301 Combinatorics
Homework 9
Due: Wednesday, November 28

1. Consider the following take-away game: There is a pile of n chips. A move consists of removing 5^k chips for some $k \geq 0$. Compute the Sprague-Grundy numbers $g(n)$ for $n \geq 0$.

   Solution: If we generate the first few Sprague-Grundy numbers then we immediately see a pattern and it makes one conjecture that $g(n) = n$ mod 2. We can verify this by induction: The base case is $n = 0$. Suppose that $n > 0$ then

   $$g(n) = mex\{g(n-1), g(n-5), \ldots\} = \\ mex\{(n-1) mod\ 2, (n-5) mod\ 2 \ldots\} = mex\{(n-1) mod\ 2\}$$

   and the claim follows.

2. In a take-away game, the set S of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where n is the number of chips remaining.

   Solution: Observe that for any finite set A, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.

3. In a take-away game, the set S of the possible numbers of chips to remove is the complement in $\{1, 2, 3, \ldots, \}$ of a finite set. Show that $g(n) \to \infty$ with n.

   Solution: We let $m = 1 + \max S$ and show that $g(n+m) > g(n)$ for all $n \geq 0$. This will prove the result. To see this, suppose ν is reachable from n i.e. $n - \nu \notin S$. But then $m + n - \nu \notin S$. This implies that $g(n+m) \geq g(n)$. But $g(n+m) \neq g(n)$ since $n + m - n \notin S$ and so n is reachable from $n + m$.