21-301 Combinatorics
Homework 8
Due: Friday, November 9

1. Suppose we 2-color the edges of K_n Red and Blue. Let r_i be the Red degree of vertex i, $i \in [n]$. (This is the degree of i in the graph induced by the Red edges).

   Show that there are exactly $\binom{n}{3} - \frac{1}{2}\sum_{i=1}^n r_i(n-1-r_i)$ mono-chromatic triangles.

   Show that if $n = 6$ then there are at least two monochromatic triangles.

   Solution: There are $r_i(n-1-r_i)$ pairs j, k such that ij is Red and ik is Blue. Each triangle that is not mono-chromatic gives rise to two such pairs. Each pair uniquely determines the triangle it is in. So half the sum is the number of non-mono-chromatic triangles.

   Now if $n = 6$, $r(5-r) \leq 6$ for integer r. Hence the sum is at most $6 \times 6 = 36$ and the number of mono-chromatic triangles is at least 20-18=2.

2. Suppose that $a, b \geq 1$ and the edges of K_{a+b} are colored Red and Blue. Show that at least one of the following exists: (i) a vertex of Red degree $\geq a$; (ii) a Blue path of length $\geq b$.
   (Hint: Assume there is no vertex of Red degree a)

   Solution: Suppose that every vertex has Red-degree $< a$. Then every vertex has Blue-degree $\geq a + b - 1 - (a-1) = b$. Now let $P = (1 = x_1, x_2, \ldots, x_s)$ be a longest Blue path starting at vertex 1. x_s has no neighbours outside P, else P can be extended to a longer path. So $s - 1 \geq b$ and P has length $\geq b$.

3. Suppose we 2-color the edges of K_n, $n \geq 3$, Red and Blue. Show that at least one of the following exists: (i) a vertex of Red degree $\geq \lfloor n/2 \rfloor - 1$; (ii) a blue triangle.

   Solution: Suppose that every vertex has Red-degree $< \lfloor n/2 \rfloor - 1$. Then every vertex has Blue-degree $\geq n - 1 - (\lfloor n/2 \rfloor - 2) = \lceil n/2 \rceil + 1$. Let u and v be joined by a Blue edge. Let X_v (resp. X_u) be the set of Blue neighbors of u (resp. v). Then $|X_u \cap X_v| \geq 2(\lceil n/2 \rceil + 1) - n \geq 2$. So,

u, v have a common Blue neighbor and this forms a Blue triangle with u, v.