21-301 Combinatorics
Homework 7
Due: Friday, November 2

1. Prove that if n is odd then for any permutation π of $[n]$, the product $P(\pi) = (1 - \pi(1))(2 - \pi(2))...(n - \pi(n))$ is necessarily even.

   Solution: Let $n = 2m+1$. Place $1, 3, 5, \ldots, 2m+1$ into boxes $B_1, B_3, B_5, \ldots, B_{2m+}$ Also, if $\pi(j) = k$ is odd, place another copy of k into box B_j. In this way we place $2m + 2$ objects into $2m + 1$ boxes. So there is a box j with two numbers j and $\pi(j) = k$ where j, k are both odd. Then $j - k$ is even and so is the product $P(\pi)$.

2. Suppose that $A \subseteq [n]$, $|A| = m$ and $2^m > mn$. Prove that it is possible to select disjoint non-empty subsets $S, T \subseteq A$ whose members have the same sum.

   Solution: Fix S and for $T \subseteq S$ let $a_T = \sum_{a \in T} a$. The integers a_T satisfy $\binom{m+1}{2} \leq a_T \leq mn - \binom{m}{2}$ and since there are 2^m of them, by the pigeon-hole principle, there exist distinct T_1, T_2 such that $a_{T_1} = a_{T_2}$. Now let $X = T_1 \cap T_2$ and $A = T_1 \setminus X$, $B = T_2 \setminus X$. Then A, B satisfy the requested conditions.

3. Given any sequence of n integers, positive or negative, not necessarily all different, show that some consecutive subsequence has the property that the sum of the members of the subsequence is a multiple of n.

   Solution Let the sequence be $x_1, x_2, \ldots, x_n$ and let $s_i = x_1 + \cdots + x_i$ mod n for $i = 1, 2, \ldots, n$. If there exists i with $s_i = 0$ then n divides $x_1 + \cdots + x_i$. Otherwise, $s_1, s_2, \ldots, s_n$ all take values in $[n - 1]$. By the pigeon-hole principle, there exist $i < j$ such that $s_i = s_j$ and then n divides $x_{i+1} + \cdots + x_j$.