21-301 Combinatorics
Homework 6
Due: Monday, October 22

1. Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$. Show that
$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$
Solution: Let π be a random permutation of $[n]$. Let $\mathcal{E}(A)$ be the event $\{\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$. Let
$$Z_i = \begin{cases} 1 & \mathcal{E}(A_i) \ occurs. \\ 0 & otherwise. \end{cases}$$
and let $Z = \sum_i Z_i$ be the number of events $\mathcal{E}(A_i)$ that occur.
Now our family is such that $Z \leq s$ for all π and so
$$\mathbf{E}(Z) = \sum_i \mathbf{E}(Z_i) = \sum_i \Pr(\mathcal{E}(A_i)) \leq s.$$
On the other hand, $A \in \mathcal{A}$ implies that $\Pr(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}}$ and the required inequality follows.

2. Let $G = (V, E)$ be an r-regular graph with n vertices. $S \subseteq V$ is a *dominating set* if $w \notin S$ implies that there exists $v \in S$ for which $\{v, w\} \in E$. Show, by the probabilistic method, that G has a dominating set of size at most $\frac{1+\ln r}{r} n$.
Solution: Let $p = \frac{\ln r}{r}$ and let S_1 be a random sub-set of V where each $v \in V$ is placed inot S, independently with probability p. Let S_2 be the set of vertices that are not adjacent to any vertex of S_1. The set $S = S_1 \cup S_2$ is a dominating set.

$$\mathbf{E}(|S|) = \mathbf{E}(|S_1|) + \mathbf{E}(|S_2|) = np + n(1-p)^{r+1} \leq np + ne^{-rp} \leq \frac{1+\ln r}{r} n.$$
So there must be a dominating set of the required size.

3. Let $G = (V, E)$ be a graph with kn vertices. Show, by the probabilistic method, that there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n$, $i = 1, 2, \ldots, k$ such that at most $|E|/k$ of the edges of G have both of their endpoints in the same part of the partition.

   Solution: Let $V_1, V_2, \ldots, V_k$ be a random partition of the vertex set. Let $e = (v, w)$ be an edge of E. Then

$$\Pr(\exists i : \ e \subseteq V_i) = \sum_{i=1}^{k} \Pr(w \in V_i \mid v \in V_i) \Pr(v \in V_i) =$$

$$\sum_{i=1}^{k} \frac{\binom{kn-2}{n-2}}{\binom{kn-1}{n-1}} \frac{1}{k} = \frac{n-1}{kn-1} < \frac{1}{k}.$$

   If Z is the number of edges of G that have both of their endpoints in the same part of the partition, then $\mathbf{E}(Z) \leq |E|/k$ and so the required partition must exist.