21-301 Combinatorics
Homework 6
Due: Monday, October 22

1. Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$. Show that
$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

2. Let $G = (V, E)$ be an r-regular graph with n vertices. $S \subseteq V$ is a *dominating set* if $w \notin S$ implies that there exists $v \in S$ for which $\{v, w\} \in E$. Show, by the probabilistic method, that G has a dominating set of size at most $\frac{1+\ln r}{r} n$.

3. Let $G = (V, E)$ be a graph with kn vertices. Show, by the probabilistic method, that there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n$, $i = 1, 2, \ldots, k$ such that at most $|E|/k$ of the edges of G have both of their endpoints in the same part of the partition.