21-301 Combinatorics Homework 5 Due: Friday, October 12

1. Let k be fixed. Show that there exists a tournament with the following property: For every set S of size k, there exists a disjoint set T of size k such that everyone in T beats everyone in S.

Solution: Let the property we want be denoted by \mathcal{A} . Let T be a random tournament on [n]. For a k-set S, let \mathcal{E}_S be the event that no T exists for this S. Then

$$\Pr(\neg \mathcal{A}) = \Pr\left(\bigcup_{S} \mathcal{E}_{S}\right) \leq \binom{n}{k} \Pr(\mathcal{E}_{[k]}).$$

If $|T| = k, T \cap [k] = \emptyset$ then we define $\mathcal{B}_{S,T}$ to be the event that everyone in T beats everyone in S. Then

$$\Pr(\mathcal{B}_{S,T}) = 2^{-k^2}$$

So, taking $T_i = \{ik + 1, \dots, (i+1)k\}$ for $i = 1, \dots, \lfloor n/k \rfloor - 1$ and using the fact that the events $\mathcal{B}_{[k],T_i}$ are independent for $i = 1, \dots, \lfloor n/k \rfloor - 1$, we see that

$$\Pr(\neg \mathcal{A}) \le \binom{n}{k} (1 - 2^{-k^2})^{n/k-2}$$

and this is < 1 for *n* sufficiently large.

2. A box has *m* drawers; Drawer i contains g_i gold coins, s_i silver coins and ℓ_i lead coins, for i = 1, 2, ..., m. Assume that one drawer is selected randomly and that two randomly selected coins from that drawer turn out to be gold. What is the probability that the chosen drawer is drawer 1?

Solution: Let G be the event that the coin is gold and let D_i be the event that drawer i is chosen. What we are asked for is

$$\Pr(D_1 \mid G) = \frac{\Pr(D_1 \cap G)}{\Pr(G)}$$

Now

$$\Pr(D_i \cap G) = \frac{1}{m} \cdot \frac{g_i(g_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}$$

and so

$$\Pr(G) = \sum_{i=1}^{m} \Pr(D_i \cap G) = \frac{1}{m} \sum_{i=1}^{m} \frac{g_i(g_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}.$$

Therefore,

$$\Pr(D_1 \mid G) = \frac{g_1(g_1 - 1)}{(g_1 + s_1 + \ell_1)(g_1 + s_1 + \ell_1 - 1)} \left(\sum_{i=1}^m \frac{g_i(g_i - 1)}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}\right)^{-1}.$$

3. Let $m = \lfloor (8/7)^{n/3} \rfloor$. Show that there exist distinct sets $A_1, A_2, \ldots, A_m \subseteq [n]$ such that for all distinct $i, j, k \in [m]$ we have $A_i \cap A_j \not\subseteq A_k$.

Let A_1, \ldots, A_m be chosen randomly from [n]. Then for fixed i, j, k we have

$$\Pr(A_i \cap A_j \subseteq A_k) = \left(\frac{7}{8}\right)^n.$$

Also,

$$\Pr(A_i = A_j) = \frac{1}{2^n}.$$

Let \mathcal{D} be the event that the sets are not distinct and let \mathcal{I} be the event that there exists i, j, k such that $A_i \cap A_j \subseteq A_k$. Then,

$$\Pr(\mathcal{D} \cup \mathcal{I}) \le {\binom{m}{2}} \frac{1}{2^n} + 3{\binom{m}{3}} \left(\frac{7}{8}\right)^n < 1.$$