21-301 Combinatorics
Homework 4
Due: Friday, October 5

1. A permutation $\pi(1), \pi(2), \ldots, \pi(n)$ is said to be 132 avoiding if there do **not** exist $i < j < k$ such that $\pi(j) > \pi(k) > \pi(i)$. Let u_n be the number of 132 avoiding permuations. Let $u_0 = 1, u_1 = 1, u_2 = 2$ and show that for $n \geq 3$,

$$u_n = \sum_{i=1}^{n} u_{i-1}u_{n-i}.$$

(Hint: Count 132 avoiding permutations with $\pi(i) = n$.)

Compare this with the recurrence for the number of ways of triangulating a polygon and get an expression for u_n.

Solution: Suppose that $\pi(i) = n$. Then to be 132 avoiding we must have $\pi(s) > \pi(t)$ for $s < i < t$. Then $\pi(1), \ldots, \pi(i-1)$ must be a 132 avoiding permutation of $[n-i+1, n-1]$ and can be any such. Similarly $\pi(i+1), \ldots, \pi(n)$ must be a 132 avoiding permutation of $[1, n-i]$ and can be any such. This explains the recurrence.

The recurrence for triangulations is $v_n = \sum_{i=0}^{n} v_k v_{n-k}$ where $v_0 = 0, v_1 = 1$. Putting $u_{-1} = 0$ we see that now $u_n = \sum_{i=0}^{n+1} u_{i-1}u_{n-i}$ and that $w_n = u_{n-1}$ satisfies $w_{n+1} = \sum_{i=0}^{n+1} w_i w_{n+1-i}$. Also, $w_0 = v_0$ and $w_1 = v_1$ and so $w_n = v_n$ for all $n \geq 0$. So, $u_n = v_{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

2. n distinguishable balls are independently and randomly numbered with m colors, each color being equally likely. What is the probability that the number of balls with color 1 is equal to the number of balls with color 2. (The answer is a sum).

Solution: Fix $0 \leq k \leq n$. The probability that there are exactly k balls with color 1 and exactly k balls with color 2 is

$$\binom{n}{k, k, n-2k} \frac{1}{m^{2k}} \left(1 - \frac{2}{m}\right)^{n-2k}.$$

So the probability in question is

$$\sum_{k=0}^{n} \binom{n}{k, k, n-2k} \frac{1}{m^{2k}} \left(1 - \frac{2}{m}\right)^{n-2k}.$$

Note that if $k > n/2$ then the summmand above is zero.

3. n indistinguishable balls are independently and randomly numbered with m colors, each coloring being equally likely. What is the probability that the number of balls with color 1 is equal to the number of balls with color 2. (The answer is a sum).

   Solution: Fix k as in question 2. If x_i is the number of balls with color i then (i) there are $\binom{m+n-1}{m-1}$ ways of coloring the balls and (ii) $\binom{m+n-2k-3}{m-3}$ ways in coloring so that $x_1 = x_2 = k$. So the probability in question is

$$\sum_{k=0}^{n} \frac{\binom{m+n-2k-3}{m-3}}{\binom{m+n-1}{m-1}}.$$