Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 3: Due Monday Octobdber 22.

Q1
Solve the following 2-person zero-sum games:

$$\begin{bmatrix} 6 & 2 & 4 \\ 5 & 2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -2 & -2 \\ 4 & 1 & 0 & -2 & -3 \end{bmatrix}$$

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.
For the second game we have the following sequence of row/column removals because of domination:

$$\text{Remove column strategy 1.} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 2.} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 3.} \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}.$$

Remove row strategy 1. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 4. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 5. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probabilty 1/2 each.

Q2

Players A and B choose integers i and j respectively from the set $\{1, 2, \ldots, n\}$ for some $n \geq 2$. Player A wins if $|i - j| = 1$. Otherwise there is no payoff. Solve the game.

Solution: Let M_n be the matrix of payoffs.

$$M_9 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now row 1 is dominated by row 3 and so let $M_n^{(1)}$ be the matrix obtained by deleting row 1 from M_n.

$$M_9^{(1)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now column 1 dominates column 3 and so we remove column 3 to obtain $M_n^{(2)}$.

$$M_9^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now column 2 dominates column 3 and so we remove column 3 to obtain $M_n^{(3)}$.

$$M_9^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now row 3 is dominated by row 5 and row 4 is dominated by row 6 and so we remove them to obtain $M_n^{(4)}$.

$$M_9^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We see that now in general, if $n \geq 8$,

$$M_n^{(4)} = \begin{bmatrix} I_2 & O_{2,n-4} \\ O_{n-5,2} & M_{n-4}^{(1)} \end{bmatrix}$$

where where I_k is the $k \times k$ identity matrix and $O_{k,l}$ is the $k \times l$ matrix of zero's.
We check that for $k \leq 7$:

$$\begin{aligned} &M_7^{(1)} \textit{ reduces to } I_4 \\ &M_6^{(1)} \textit{ reduces to } I_4 \\ &M_5^{(1)} \textit{ reduces to } I_3 \\ &M_4^{(1)} \textit{ reduces to } I_2 \\ &M_3^{(1)} \textit{ reduces to } I_2 \\ &M_2^{(1)} \textit{ reduces to } I_2 \\ &M_1^{(1)} \textit{ reduces to } I_1 \end{aligned}$$

It follows inductively, that dominance reduces the matrix $M_n^{(1)}$ to I_{2k+l} where $k = \lfloor n/4 \rfloor$ and $l = 1 + 1_{n \neq 1 \mod 4}$. A game with playoff matrix I_s has a solution where each player plays each choice with probability s^{-1}.

Q3

Player B chooses a number $j \in \{1, 2, \ldots, n\}$ and A tries to guess what it is. If A guesses correctly then A wins 1. If A guesses too high then A loses 1. If A guesses too low there is no payoff. Solve the game.

Solution: Let A_n be the matrix of the game. For example,

$$A_9 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

The matrix A_n is non-singular. These games are discussed in my notes, on the last page. Let $\mathbf{1}_n$ be the $n \times 1$ matrix $[1, 1, \ldots, 1]^T$.
Now the solution to $A_n y = \mathbf{1}_n$ is given by $y_j = 2^{j-1}$ and is non-negative. Thus $y_1 + \cdots + y_n = 2^n - 1$. So we put $q_j = \frac{2^{j-1}}{2^n - 1}$.
The solution to $x^T A = \mathbf{1}_n^T$ is given by $x_i = 2^{n-i}$ and is non-negative. Thus $x_1 + \cdots + x_n = 2^n - 1$. So we put $p_i = \frac{2^{n-i}}{2^n - 1}$.
The vectors p, q solve the game. This follows from the notes. One can check that they solve the dual pair of linear programs associated with the game.