21-301 Combinatorics
Homework 2
Due: Friday, September 14

1. Use induction to show that
$$\binom{n-1}{k} = \binom{n}{k} - \binom{n}{k-1} + \cdots \pm \binom{n}{0}.$$
Solution We use induction on k for a fixed n.
Base Case: $k = 0$. This is trivial, $\binom{k}{0} = \binom{k-1}{0}$.
Inductive Step: Suppose that the identity is true for some $k \geq 0$. Then
$$\begin{aligned}\binom{n}{k+1} - \binom{n}{k} + \cdots \pm \binom{n}{0} &= \binom{n}{k+1} - \left(\binom{n}{k} - \cdots \pm \binom{n}{0}\right) \\ &= \binom{n}{k+1} - \binom{n-1}{k} \qquad \textit{Induction} \\ &= \binom{n-1}{k+1}. \qquad \textit{Pascal's Triangle}\end{aligned}$$

2. In how many ways can $3n$ distinguishable balls $b_1, b_2, \ldots, b_{3n}$ be placed in boxes $B_1, B_2, \ldots, B_n$ so that (i) each box contains three balls and (ii) there does not exist i such that box B_i contains balls $b_{3i-2}, b_{3i-1}, b_{3i}$?
Solution: Let A denote the set of allocations of $3n$ balls to n boxes, 3 to a box and let A_i denote the set of allocations in box i gets balls $b_{3i-2}, b_{3i-1}, b_{3i}$. Then
$$|A_S| = \frac{3(n-|S|)!}{3!^{n-|S|}}$$
and so the number of scrambled allocations is
$$\begin{aligned}\left|\bigcap_{i=1}^{n} \bar{A}_i\right| &= \sum_{S \subseteq [n]} (-1)^{|S|} |A_S| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} \frac{3(n-|S|)!}{3!^{n-|S|}} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{3(n-k)!}{3!^{n-k}}.\end{aligned}$$

3. Find an expression for the size of the set
$$\{(x_1, x_2 \ldots, x_m)\} \in Z^m :\ x_1 + x_2 + \cdots + x_m = n \textit{ and } 1 \leq x_j \leq a \textit{ for } j = 1, 2, \ldots, m\}.$$
[You should use Inclusion-Exclusion and expect to have your answer as a sum.]
Solution: Let
$$A = \{(x_1, x_2 \ldots, x_m)\} \in Z^m :\ x_1 + x_2 + \cdots + x_m = n \textit{ and } 1 \leq x_j \textit{ for } j = 1, 2, \ldots, m\}.$$

Then let

$$A_i = \{x \in A : \ x_i \geq a+1\}.$$

Now,

$$A_S = \{(x_1, x_2 \ldots, x_m)\} \in Z^m : \ x_1 + x_2 + \cdots + x_m = n \\ \qquad and\ 1 \leq x_j \ for\ j \notin S,\ a+1 \leq x_j \ for\ j \in S\}.$$

So,

$$|A_S| = \binom{n-1-a|S|}{m-1}.$$

Then we must compute

$$\begin{aligned} \left|\bigcap_{i=1}^{m} \bar{A}_i\right| &= \sum_{S \subseteq [m]} (-1)^{|S|} |A_S| \\ &= \sum_{S \subseteq [m]} (-1)^{|S|} \binom{n-1-a|S|}{m-1} \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n-1-ak}{m-1}. \end{aligned}$$