21-301 Combinatorics
Homework 1: Solutions
Due: Wednesday, September 7

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

satisfy $x_1 \geq 3$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 6$ and $x_5 \geq 0$?

Solution Let

$$y_1 = x_1 - 3, \qquad y_2 = x_2 - 10, \qquad y_3 = x_3 + 3, \qquad y_4 = x_4 - 6, \qquad y_5 = x_5.$$

An integral solution of $x_1+x_2+x_3+x_4+x_5 = 100$ such that $x_1 \geq 3$, $x_2 \geq 10$, $x_3 \geq -3$, $x_4 \geq 6$ and $x_5 \geq 0$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 + y_5 = 84$ such that $y_1, \dots, y_5 \geq 0$. From a result in class,

$$|\{(y_1, y_2, y_3, y_4, y_5) : y_1, \dots, y_5 \in Z_+ \text{ and } y_1 + \cdots + y_5 = 84\}| = \binom{84+5-1}{5-1} = \binom{88}{4}.$$

2. Prove the following equality using a *combinatorial* argument

$$\sum_{i=3}^{n} \binom{i}{3}\binom{n}{i} = \binom{n}{3}2^{n-3}.$$

Solution Consider the set

$$\mathcal{S} = \{(A, \{x, y, z\}) : A \subseteq [n] \quad \text{and} \quad \{x, y, z\} \subseteq A\}.$$

In words, $\mathcal{S}$ is the set of all ordered pairs consisting of a subset of $[n]$ and three elements of that set. We count the elements of $\mathcal{S}$ is two ways.

First we count with respect to the elements x, y, z. There are $\binom{n}{3}$ choices for x, y, z. Once x, y, z are fixed, $A \setminus \{x, y, z$ can be any subset of $[n] \setminus \{\{x, y, z\}\}$. There are 2^{n-3} such sets. Therefore, we have

$$|\mathcal{S}| = n2^{n-3}.$$

Now we count with respect to the set A. For $i = 1, 2, \dots, n$ let

$$\mathcal{S}_i = \{(A, \{x, y, z\}) \in \mathcal{S} : |A| = i\}.$$

These sets form a partition of $\mathcal{S}$. There are $\binom{n}{i}$ choices for the set A in an ordered pair in $\mathcal{S}_i$. Once this set is fixed there are $\binom{i}{3}$ choices for x, y. Therefore

$$|\mathcal{S}_i| = \binom{n}{i}\binom{i}{3},$$

and

$$|\mathcal{S}| = \sum_{i=1}^{n} |\mathcal{S}_i| = \sum_{i=1}^{n} \binom{n}{i}\binom{i}{3}.$$

The result is given by noting that the two expressions for $|\mathcal{S}|$ are equal.

3. A sequence $a_1a_2\cdots a_m$ where $a_i \in [n]$ is *s-spaced out* if $a_{i+1} \geq a_i + s$ for $1 \leq i < m$. s is a non-negative integer. Show that the number of such sequences is $\binom{m+n-s(m-1)-1}{m}$.

   Solution Let $x_i = a_i - a_{i-1}$ for $i = 2, 3, \ldots, m$ and $x_1 = a_1$ and $x_{m+1} = n - a_m$. Then we have

$$x_1 + x_2 + \cdots + x_{m+1} = n$$

   and $x_1 \geq 1, x_2, \ldots, x_m \geq s, x_{m+1} \geq 0$.

   Furthermore, there is a bijection between the x's and our set of functions. The number of x's is given by the binomial coefficient.