21-301 Combinatorics
Homework 10
Due: Wednesday, December 5

1. How many ways are there to 3-color an $n \times n$ chessboard when n is odd. The group G is the usual 8 element group e, a, b, c, p, q, r, s.

   Solution: All we need do is compute the number of cycles in the permutations as applied to the chessboard. Let $n = 2m + 1$. In the table, $r \times s$ is short for r cycles of length s.

Permutation	Number of cycles
e	$n^2 \times 1$
a	$m(m+1) \times 4 + 1 \times 1$
b	$2m(m+1) \times 2 + 1 \times 1$
c	$m(m+1) \times 4 + 1 \times 1$
p	$mn \times 2 + n \times 1$
q	$mn \times 2 + n \times 1$
r	$mn \times 2 + n \times 1$
s	$mn \times 2 + n \times 1$

   Thus the number of colorings is

$$\frac{1}{8}(3^{n^2} + 2 \times 3^{m(m+1)+1} + 3^{2m(m+1)+1} + 4 \times 3^{mn+n}).$$

2. A necklace is made of 8 beads strung together in a cycle. Find the pattern inventory for the two colourings of the necklace. when the group G is the group of rotations.

   Solution: We can borrow from the solution to Q3.

$$PI = \frac{1}{8}((b+w)^8 + 4(b^8 + w^8) + 2(b^4 + w^4)^2 + (b^2 + w^2)^4).$$

3. A necklace is made of 8 beads strung together in a cycle. Find the pattern inventory for the two colourings of the necklace when the group G is the dihedral group D_8.

The *dihedral* group D_n is the group of symmetries of a regular n-gon under rotations $R_0, R_1, \ldots, R_{n-1}$ and reflections $S_0, S_1 \ldots, S_{n_1}$, with composition given by the following formulas:

$$R_i\,R_j = R_{i+j}, \quad R_i\,S_j = S_{i+j}, \quad S_i\,R_j = S_{i-j}, \quad S_i\,S_j = R_{i-j}.$$

Solution: The group of symmetries consists of

(i) $e = r_0, r_1, \ldots, r_7$ where r_i is *rotation through* $i\pi/4$,

(ii) s_1, s_2, s_3, s_4 where s_i is *a rotation about an axis joining two opposite vertices*,

(iii) t_1, t_2, t_3, t_4 where t_i is *a rotation about an axis joining the midpoints of two opposite edges.*

π	e	r_1	r_2	r_3	r_4	r_5	r_6	r_7	s_i	t_i
$\mathrm{ct}(\pi)$	x_1^8	x_8	x_4^2	x_8	x_2^4	x_8	x_4^2	x_8	$x_1^2x_2^3$	x_2^4

Thus

$$PI = \frac{1}{16}((b+w)^8+4(b^8+w^8)+2(b^4+w^4)^2+4(b+w)^2(b^2+w^2)^3+5(b^2+w^2)^4).$$