9\19\2007

$$a_n - 3a_{n-1} = n^2$$

$$a(x)(1-3x) - 1 = \sum_{n=1}^{\infty} n^2 x^n$$

Tool Kit

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n$$

$$\frac{2}{(1-x)^3} = 2 + (3\times 2)x + (4\times 3)x^2 + \cdots + (n+2)(n+1)x^n$$

$$n^2 = n(n-1) + n$$

$$\sum_{n=1}^{\infty} n^2 x^2 = \sum_{n=\cancel{1}\,2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} n x^n$$

$$\sum_{n=2}^{\infty} n(n-1)x^n = 2x^2 + (3\times 2)x^3 + \dots = \frac{2x^2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2}$$

$$a(x)(1-3x) - 1 = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$a(x) = \frac{1}{1-3x} + \frac{2x^2}{(1-3x)(1-x)^3} + \frac{x}{(1-x)^2(1-3x)}$$

$$a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x}$$

$$\downarrow$$

$$-\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2} \times 3^n$$

Product of Generating Function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\,b(x) = (a_0 + a_1 x + a_2 x^2 + \cdots) \times (b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\,x^2 + \cdots$$

$$a(x)\, b(x) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Derangements

$d_n = \#$ of derangements of size n.

choose the fixed points

$$n! = \sum_{k=0}^{n} \underbrace{\binom{n}{k} d_{n-k}}$$

derange the rest

\# permutations

\# of permutations with k fixed points.

i is a fixed point if $\pi(i) = i$.

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} \qquad n \geq 0$$

$$1 = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{d_{n-k}}{(n-k)!} \qquad n \geq 0$$

$\times x^n$ & sum

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$$a(x)b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \, b_{n-k} \right) x^n$$

$\times x^n$ & sum

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\,\frac{d_{n-k}}{(n-k)!}\right) x^n$$

$$a(x)b(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\, b_{n-k}\right) x^n$$

$$\frac{1}{1-x} = \underbrace{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)}_{e^x}\underbrace{\left(\sum_{n=0}^{\infty} \frac{d_n}{n!} x^n\right)}_{d(x)}$$

$$= e^x \times d(x)$$

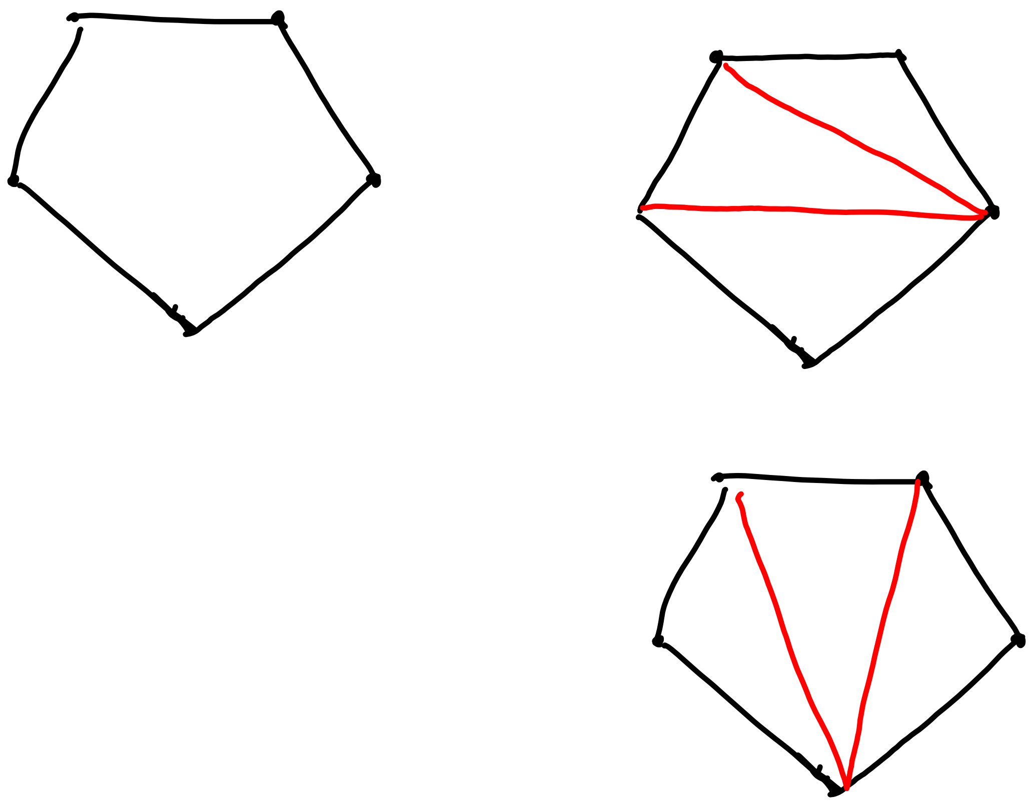
$$d(x) = \frac{e^{-x}}{1-x}$$

$$d(x) = \frac{e^{-x}}{1-x}$$

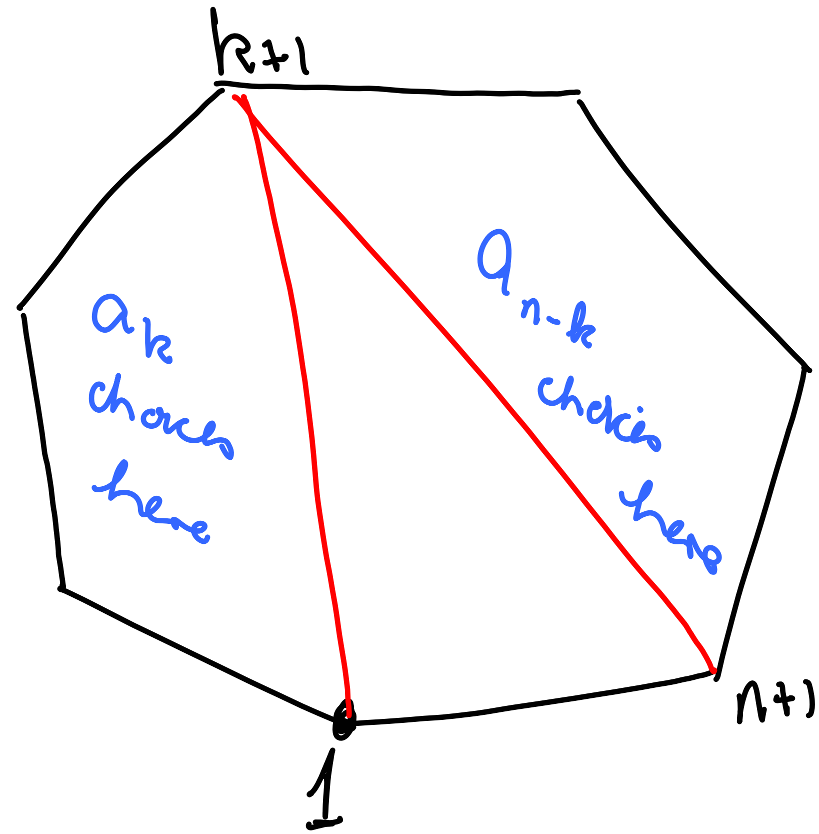
$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \right) \left(\sum_{n=0}^{\infty} x^n \right)$$

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \times 1$$

Triangulation of an n-gon



a_n = # of triangulations P_{n+1}



$a_0 = a_1 = 1$

$$a_n = \sum_{k=0}^{n} a_k a_{n-k}$$