

9/19/2007

$$a_n - 3a_{n-1} = n^2$$



$$a(x)(1-3x) - 1 = \sum_{n=1}^{\infty} n^2 x^n$$

Tool kit

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n$$

$$\frac{2}{(1-x)^3} = 2 + (3 \times 2)x + (4 \times 3)x^2 + \dots + (n+2)(n+1)x^n$$

$$n^2 = n(n-1) + n$$

$$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=2}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n$$

$\parallel$ $\parallel$
 $2x^2 + (3 \times 2)x^3 + \dots$ $x + 2x^2 + 3x^3 + \dots$
 $\parallel$ $\parallel$
 $\frac{2x^2}{(1-x)^3}$ $\frac{x}{(1-x)^2}$

$$a(x)(1-3x) - 1 = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$a(x) = \frac{1}{1-3x} + \frac{2x^2}{(1-3x)(1-x)^3} + \frac{x}{(1-x)^2(1-3x)}$$

$$a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x}$$

$$\downarrow$$

$$-\frac{1}{2} \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2} \sum_{n=0}^{\infty} 3^n x^n$$

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2} \cdot 3^n$$

Product of Generating Function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)b(x) = (a_0 + a_1x + a_2x^2 + \dots) \times (b_0 + b_1x + b_2x^2 + \dots)$$

$$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

$$a(x) b(x) = \sum_{n=0}^{\infty} C_n x^n$$

where

$$C_n = \sum_{k=0}^n a_k b_{n-k}$$

Derangements

$d_n = \#$ of derangements of size n .

$$n! = \sum_{k=0}^n \underbrace{\binom{n}{k} d_{n-k}}_{\substack{\# \text{ of permutations with } k \\ \text{fixed points.}}}$$

$n!$ is labeled with a red arrow pointing to it from the text "# permutations".

$\binom{n}{k}$ is labeled with a red arrow pointing to it from the text "# of permutations with k fixed points."

d_{n-k} is labeled with an orange arrow pointing to it from the text "derange the rest".

The entire term $\binom{n}{k} d_{n-k}$ is underlined in red.

An orange arrow points from the text "choose the fixed points" to the binomial coefficient $\binom{n}{k}$.

i is a fixed point if $\pi(i) = i$.

$$n! = \sum_{k=0}^n \binom{n}{k} d_{n-k} \quad n \geq 0$$

$$1 = \sum_{k=0}^n \frac{1}{k!} \cdot \frac{d_{n-k}}{(n-k)!} \quad n \geq 0$$

x x^n & sum

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n$$

$a(x)b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) x^n$

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n \quad \times x^n \& \text{sum}$$

$$a(x)b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) x^n$$

$$\frac{1}{1-x} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{d_n}{n!} x^n \right)$$

$$= e^x \times d(x)$$

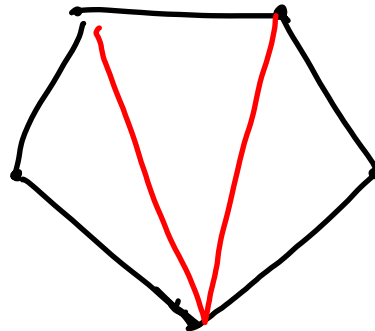
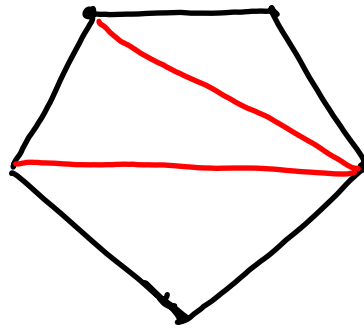
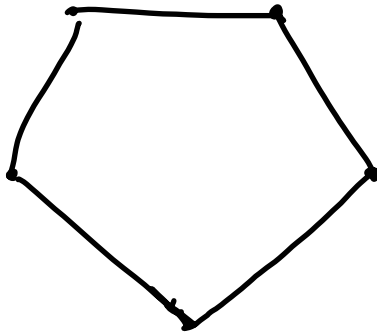
$$d(x) = \frac{e^{-x}}{1-x}$$

$$d(x) = \frac{e^{-x}}{1-x}$$

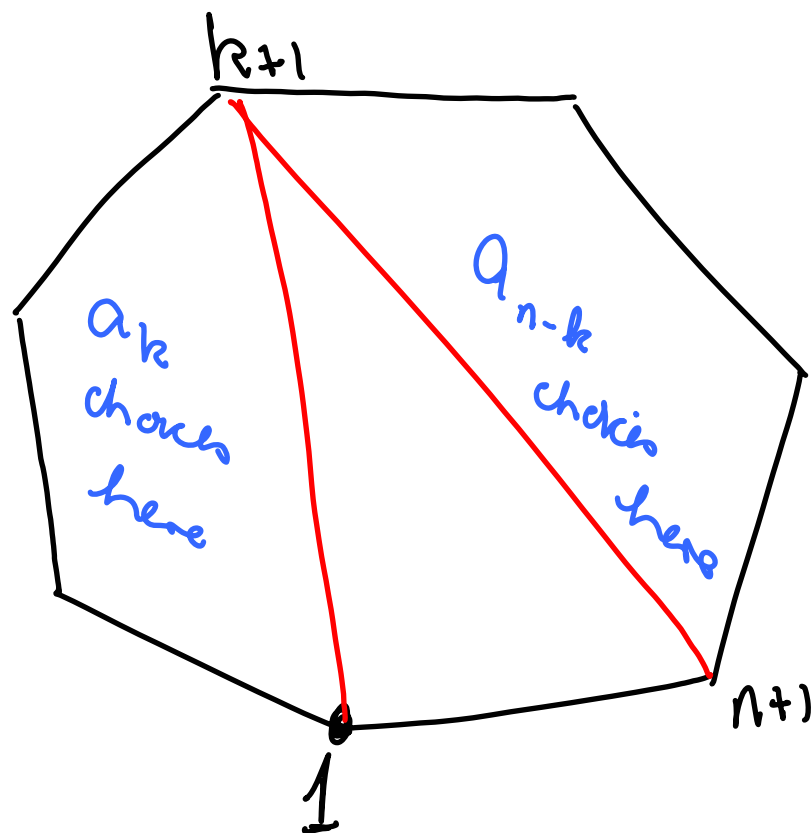
$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \right) \left(\sum_{n=0}^{\infty} x^n \right)$$

$$\frac{d_n}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!} \times 1$$

Triangulation of an n -gon



$a_n = \# \text{ of triangulations of } P_{n+1}$



$$a_0 = a_1 = 1$$

$$a_n = \sum_{k=0}^n a_k a_{n-k}$$