9/14/2007

"Unknown" sequence $a_0, a_1, a_2, \ldots a_n \ldots$

We know the first few and we also know that

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$$

Problem: Solve this set of equations

We know $a_0, a_1, \ldots, a_{k-1}$

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2$$

$a_0 = a_1 = 1$

$a_2 = 1 + 1 = 2$

$a_3 = 2 + 1 = 3$

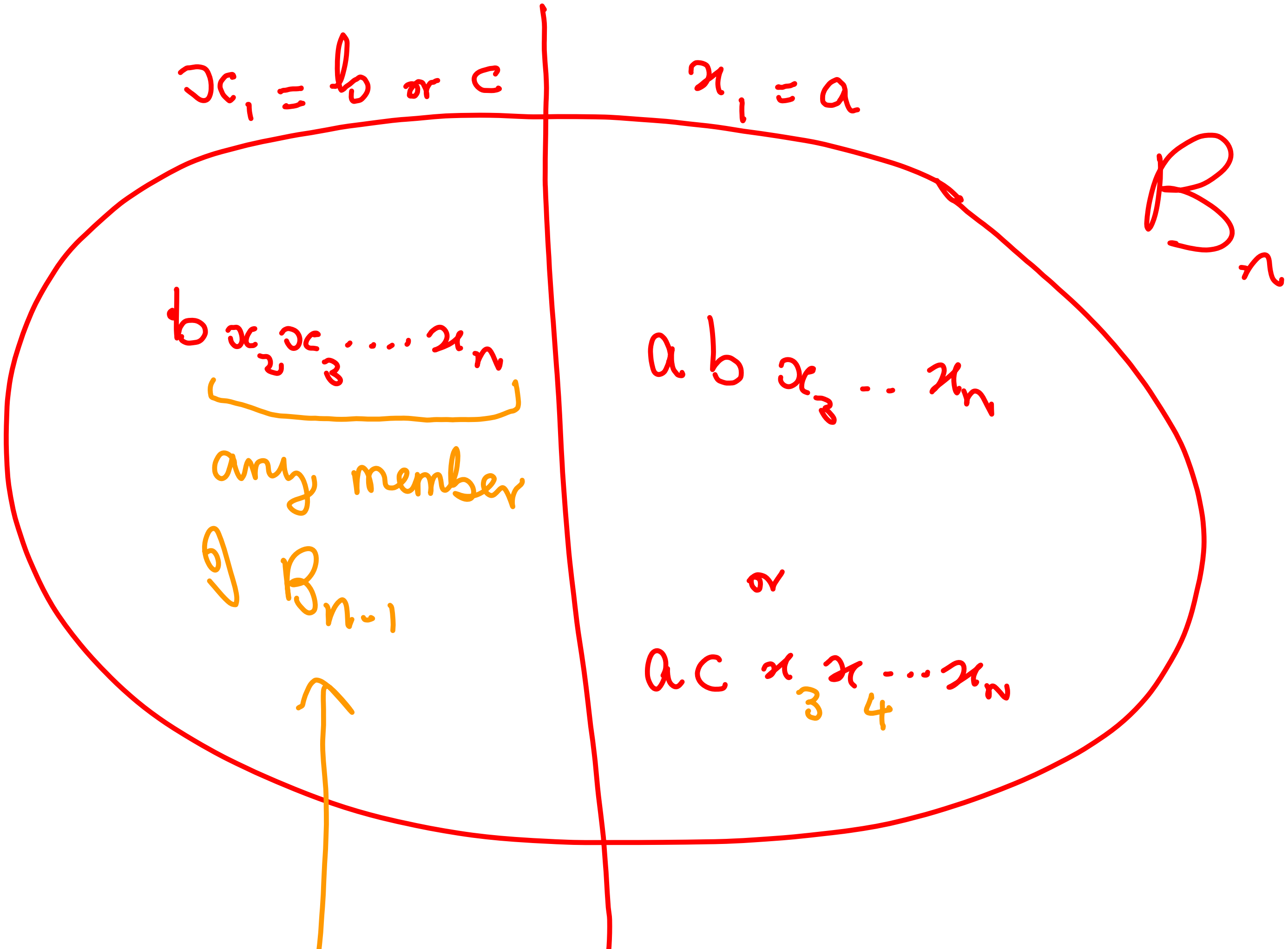
$\vdots$

$$b_n = |B_n| = \{x \in \{a, b, c\}^n : \text{ no } aa\}$$

$b_1 = 3$ a b c

$b_2 = 8$ ab ac ba bb . . . . .

$$b_n = 2b_{n-1} + 2b_{n-2} \quad ???$$



$$b_n = \overbrace{2b_{n-1}} + \overbrace{2b_{n-2}}$$

Towers of Hanoi

H_n = minimum number of moves needed.

$$= H_{n-1} + 1 + H_{n-1}$$

move n-1 rings above largest to peg 2.

move large ring to 3

move rings on peg 2 → peg 3.

A has \$n.

Each day he can buy either

a Bun (B) costs \$1

Ice Cream (I) costs \$2

Pastry (P) costs \$2

Buying sequence looks like BBBIIPIIBI...

u_n = # sequences.

$u_0 = 1$; Λ empty string

$u_1 = 1$ B

$$U_n = U_{n-1} + 2U_{n-2}$$

start with B

start with I or P.

If $a_0, a_1, \ldots a_n, \ldots$ is

a sequence of real numbers, then

its (ordinary) generation function is

the function

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

$$a_n = [x^n]\, a(x)$$

Binomial theorem

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n$$

$$= \sum_{n=0}^{\cancel{m}\,\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$$

m can be any real number.

Solution of linear recurrence

$$a_n - 2a_{n-1} = 1 \qquad n \geq 1$$

$$a_0 = 1$$

multiply each equation by x^n and sum

$$\sum_{n=1}^{\infty} (a_n - 2a_{n-1}) x^n = \sum_{n=1}^{\infty} x^n$$

$$\underbrace{\sum_{n=1}^{\infty} a_n x^n}_{a(x) - a_0} - 2 \underbrace{\sum_{n=1}^{\infty} a_{n-1} x^n}_{\substack{a_0 x + a_1 x^2 + \cdots \\ x a(x)}} = \frac{x}{1-x}$$

$$\underbrace{\sum_{n=1}^{\infty} a_n x^n}_{a(x) - a_0} - 2 \underbrace{\sum_{n=1}^{\infty} a_{n-1} x^n}_{\substack{a_0 x + a_1 x^2 + \cdots \\ x\,a(x)}} = \frac{x}{1-x}$$

$$a(x) - 1 - 2x\,a(x) = \frac{x}{1-x}$$

$$a(x)\left[1 - 2x\right] = 1 + \frac{x}{1-x} = \frac{1}{1-x}$$

$$a(x) = \frac{1}{(1-x)(1-2x)}$$

$$a(x) = \frac{1}{(1-x)(1-2x)}$$

$$= \frac{A}{1-x} + \frac{B}{1-2x}$$

So, $\forall x$

$$A(1-2x) + B(1-x) = 1$$

$x = 1$: $-A = 1$

$x = \frac{1}{2}$: $B = 2$

$$a(x) = -\frac{1}{1-x} + \frac{1}{1-2x}$$

$$a(x) = -\frac{1}{1-x} + \frac{2}{1-2x}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[-1 + 2^{n+1}\right]}_{a_n} x^n$$