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$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$x \in A$

$$(1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) = \begin{cases} 1 & x \in \bigcap \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \left| \bigcap_{i \in [N]} \bar{A}_i \right| &= \sum_{x \in A} (1 - \theta_{x,1})(1 - \theta_{x,2}) \cdots (1 - \theta_{x,N}) \\ &= \sum_{x \in A} \sum_S (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \end{aligned}$$

places where we
choose the -1 term

Now change order of
summation:

$$= \sum_{x \in A} \sum_S (-1)^{|S|} \prod_{i \in S} \theta_{x,i}$$

$$= \sum_S (-1)^{|S|} \underbrace{\sum_{x \in A} \prod_{i \in S} \theta_{x,i}}_{1 \text{ iff } x \in \bigcap_{i \in S} A_i}$$

$$= \sum_S (-1)^{|S|} |A_S|.$$

Euler's function $\phi(n)$ Totient

$\phi(n)$ = number of integers in $\{2, 3, \dots, n\}$

that have no common factors with n .

$$\phi(20) = 8: 1, 3, 7, 9, 11, 13, 17, 19$$

$$\begin{aligned}\phi(20) &= 20 \times \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) \\ &= 8\end{aligned}$$

2, 5 are
prime factors
of 20

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

where $p_1, p_2, \dots, p_k > 1$ are the prime factors of n .

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right) ?$$

Inclusion-Exclusion:

$$A = [n]$$

Remove x if it has a factor p_1 or p_2 or $\dots$ or p_k

To use inclusion-exclusion we must
define the sets to exclude: A_i .

$$A_i = \{x \in [n] : p_i \mid x\}$$

$$\phi(n) = \left| \bigcap_{i=1}^k \overline{A_i} \right|$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} |A_S|$$

$$= \sum_S (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

$$|A_1| = \frac{n}{p_1}$$

$$|A_{\{1,2\}}| = \frac{n}{p_1 p_2}$$

divisor

of functions $f: [n] \xrightarrow{\text{onto}} [m]$

$$\# \text{ functions} = m^n$$

$$\# \text{ 1-1 functions} = m(m-1) \dots (m-n+1)$$

Onto functions = ??

$$F(n, m) = \{ \text{onto functions} \}$$

$$= \bigcap_{i=1}^m \overline{A_i}$$

$$A_i = \{ f : f \text{ misses } i \}$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} |A_S|$$

fns that miss S
 $= \# f: [n] \rightarrow [m] \setminus S$
 $= (m - |S|)^n$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n$$

$$= \sum_{s=0}^m \binom{m}{s} (-1)^s (m-s)^n$$

$$[= 0 \cdot \checkmark \quad m > n !!!]$$