9/07/2007

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |A| - |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$- |A_1 \cap A_2 \cap A_3|$$

General Case

N sets $A_1, A_2, \ldots, A_N \subseteq A$

$$A_S = \bigcap_{i \in S} A_i$$

$$A_{\{3,5,8\}} = A_3 \cap A_5 \cap A_8$$

$$A_\emptyset = A.$$

Two sorts of subsets:

Subsets of A

Subsets of $[N]$ — index sets

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_N| =$$

$$|A|$$

$$- |A_1| - |A_2| - |A_3| - \cdots - -$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{N-1} \cap A_N|$$

$$- |A_1 \cap A_2 \cap A_3| - \cdots$$

$$+$$

How many integers in $[10^6]$ are <u>not</u> divisible by 8, 12 or 20.

$A_1 = \{x : 8 \mid x\}$ $\quad$ $A_2 = \{x : 12 \mid x\}$

$A_3 = \{x : 20 \mid x\}$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |A| - |A_1| - |A_2| - |A_3|$$

$$10^6 - 125{,}000 - 83{,}333 - 50{,}000$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$+ 41666 + 25000 + 16{,}666$$

$$- |A_1 \cap A_2 \cap A_3|$$

$$8333$$

Derangements

$n!$ permutations of $\{1, 2, \ldots, n\}$

Permutation is function $\pi : [n] \to [n]$

$\underline{n=6}$							
	i	1	2	3	4	5	6
	$\pi(i)$	2	3	4	5	6	1
		3	6	1	4	2	5

$\pi(i) \neq i$ for all i : Derangement.

not a derangement.

$$A = \{\text{permutations}\}$$

$$A_i = \{\pi : \pi(i) = i\}$$

$$\underbrace{|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|}_{\text{derangements}} = \sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$$

$$|A_S| = ?$$

$$|A_\emptyset| = n!$$

$$|A_1| = \{\pi : \pi(1) = 1\}$$

i	1	2	3	4	$\cdots$	n
$\pi(i)$	1	*	*	*	-	*

$$|A_1| = |A_2| = \dots = |A_n| = (n-1)!$$

$|A_{\{1,3\}}|$

i	1	2	3	4	5	. . .	n
$\pi(i)$	$\underline{1}$	*	3	*	* .		*

$|A_{\{1,3\}}| = (n-2)!$

$|A_S| = (n-|S|)!$

derangements

$$= \sum_{S \subseteq [n]} (-1)^{|S|} (n-|S|)!$$

$$= \sum_{k=0}^{n} \sum_{S:|S|=k} (-1)^k (n-k)!$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = \sum_{k=0}^{n} \frac{n!}{k!} (-1)^k$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} .$$