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Integers a, b

binary expansion

$a = a_m a_{m-1} \cdot a_i \cdot\cdot\ a_2 a_1 a_0$

$b = b_m b_{m-1} \cdot b_i \cdot\ b_2 b_1 b_0$

$a \oplus b = \quad a_i \oplus b_i$

exclusive OR

$12 \oplus 23$

12	0 1 1 0 0
23	1 0 1 1 1
27	1 1 0 1 1

Theorem

$g_i \equiv$ grundy function for game G_i

$g \equiv$ " " " " $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$

$x = (x_1, x_2, \ldots, x_p)$ is position in G

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

We can assume $p=2$ and use induction on p.

Suppose proved true for $p=2$.

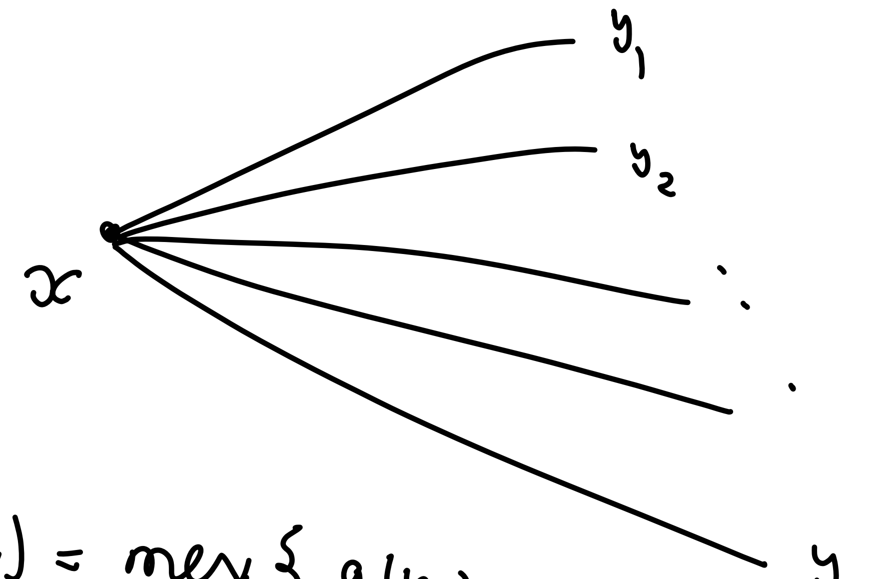
Now use induction on p

$H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$

$G = H \oplus G_p$

$g = (\text{grundy } H) \oplus g_p$ —— 2-case

$= g_1 \oplus \cdots + g_{p-1} \oplus g_p$ —— induction



$$g(x) = \text{mex}\{g(y_1), g(y_2), \ldots, g(y_r)\}$$

Need to show:

A1: if $x \in X$ and $g(x) = b > a$
then $\exists\, x' \in N^+(x)$ s.t. $g(x') = a$

A2: if $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$
then $g(x') \neq b$.

$$g(x) = b > a$$

$$a = d \oplus b = \underbrace{d \oplus g_1(x_1)} \oplus g_2(x_2)$$

Suppose this is $< g_1(x_1)$

then $\exists$ move in G_1 to $d \oplus g_1(x_1)$

$\Rightarrow$ move to a in G

Need to show either

$$d \oplus g_1(x_1) < g_1(x_1)$$

or

$$d \oplus g_2(x_2) < g_2(x_2)$$

Suppose $2^{k-1} \leq d < 2^k$

$d_k = 1$ & $d_l = 0,\ l > k.$

$$d_k = a_k \oplus b_k$$

(i) $a_k = 0,\ b_k = 1$ or (ii) $\underbrace{a_k = 1,\ b_k = 0}_{\text{Not possible}}$

Not possible
$a < b$
and $a_l = b_l,\ l > k.$

$b_k = 1$

$\Rightarrow$ $g_1(x_1)$ has a 1 in pos. k
or $g_2(x_2)$ has a 1 in pos. k

Assume $g_1(x_1)$ has a 1 in position k

$d \oplus g_1(x_1)$

							k				
d	0	0	0	0	0	0	1				
$g_1(x_1)$	*	*	*	*	*	*	1				
$d \oplus g_1(x)$	*	*	*	*	*	*	0	-	-	-	-

A1 is done.

A2:

Suppose

$$b = g_1(x_1) \oplus g_2(x_2)$$
$$= g_1(x_1') \oplus g_2(x_2)$$

$$\Longrightarrow \quad g_1(x_1) = g_1(x_1')$$

Contradiction $g_1(x_1) > g_1(x_1')$