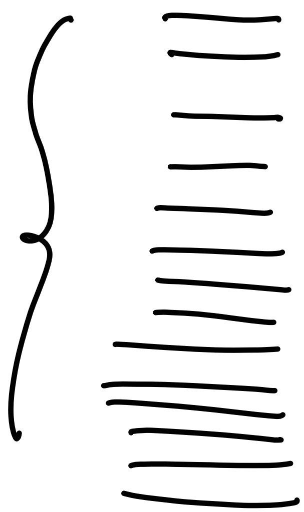
11\14\2007

# Combinatorial Games

Two players. A & B. A starts.

$n$ chips



A moves

B moves

A moves

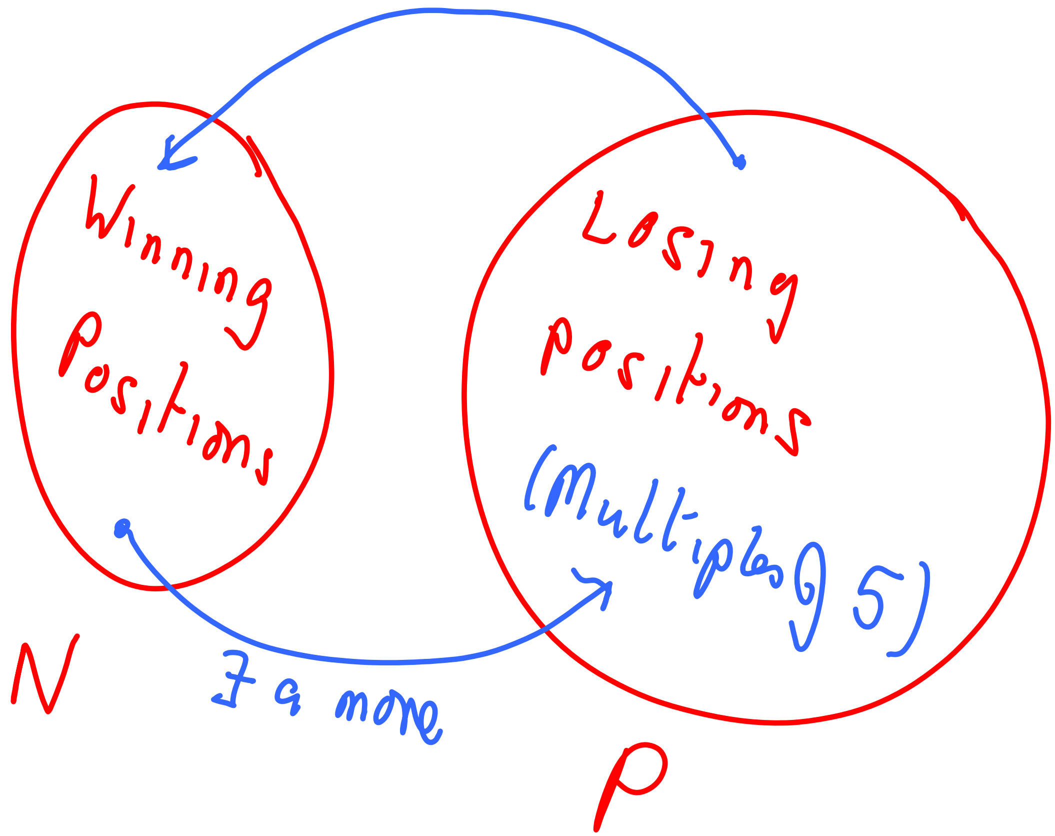
⋮

Until <u>next</u> player cannot move → This player loses.

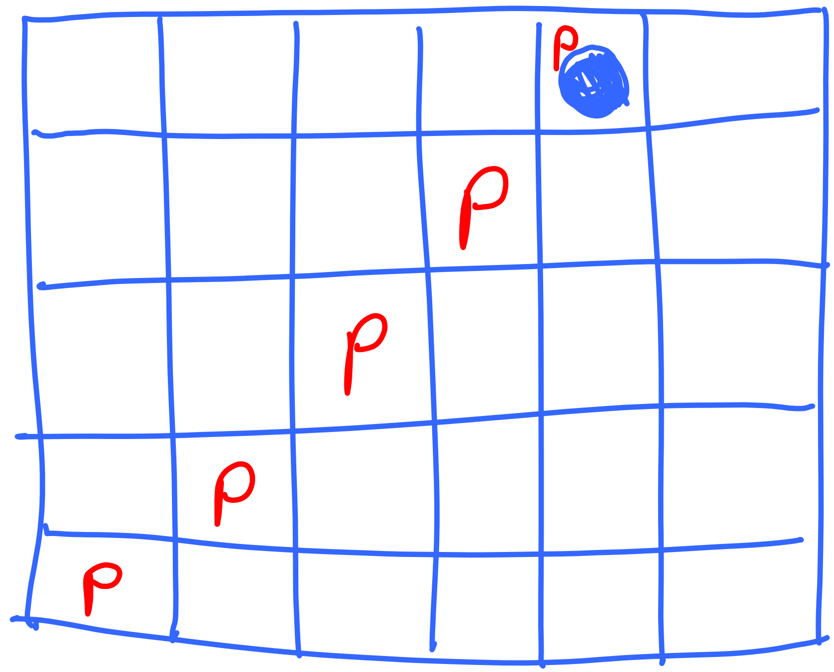
Move:

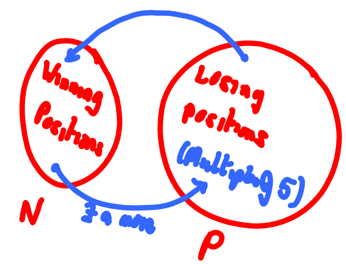
Delete 1, 2, 3 or 4 chips.

$n = 23$

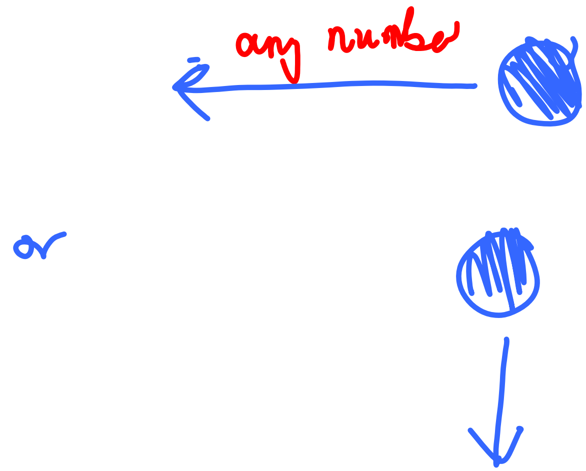

Winning positions
Losing positions
(Multiples of 5)
N
∃ a move
P

2/



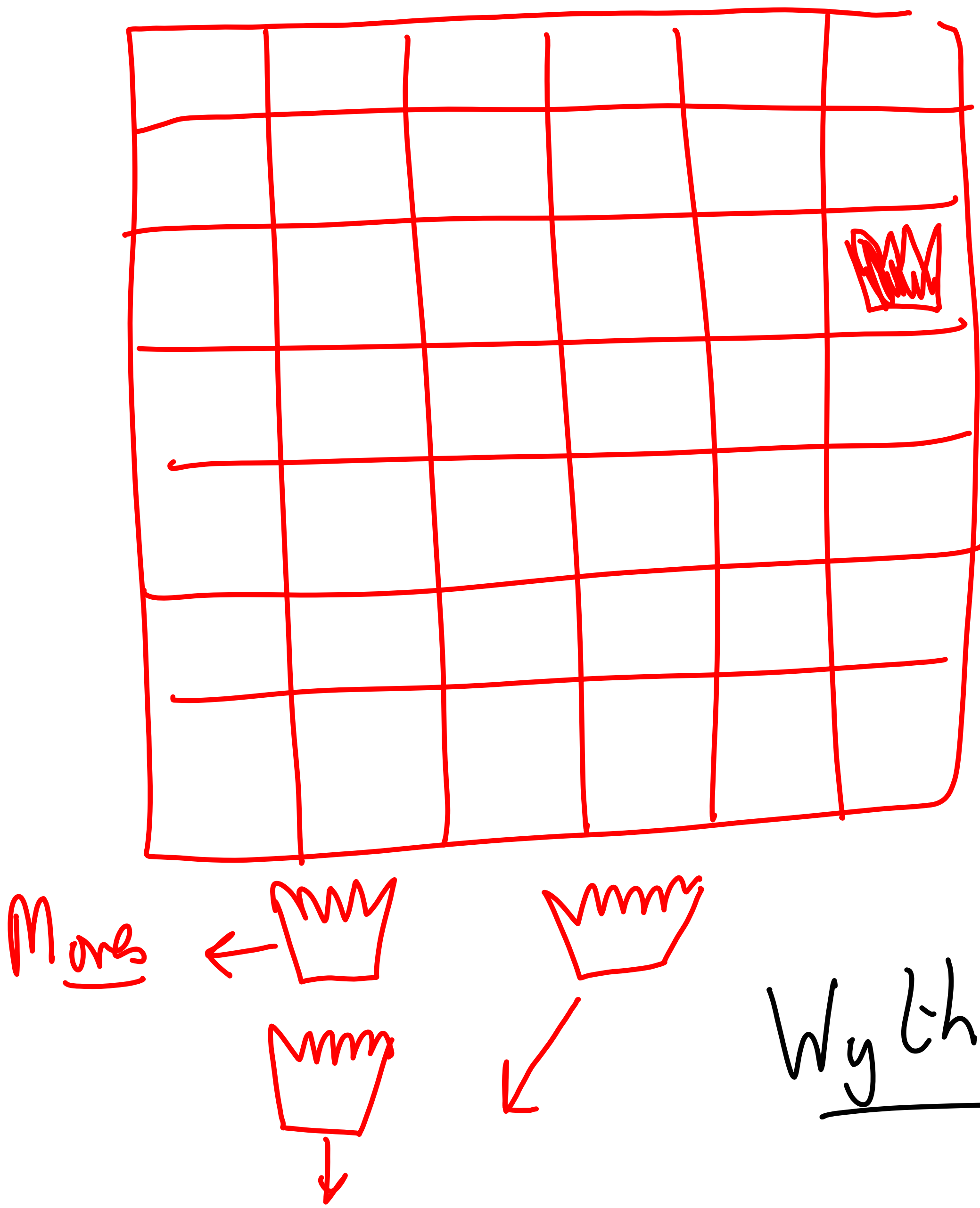




Move:




Really 2 piles of chips

2 pile Nim




Wythoff's Nim

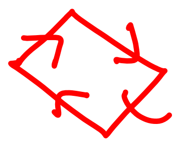
A games is a digraph $(X, A)$

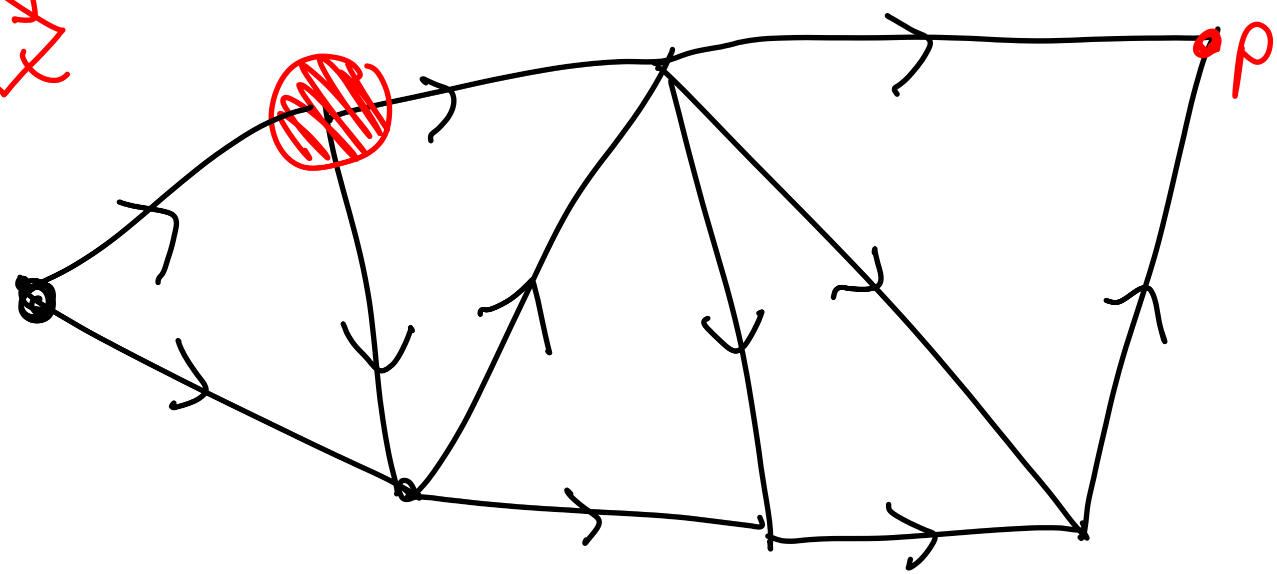
vertices (Board)

allowed moves

(i) Finite $|X| < \infty$

(ii) D has no directed cycles






Each vertex represents a possible position of game.

Move : move token here $\longrightarrow$ there

**Game 1:** $X = \{0, 1, 2, \ldots, n\}$

$A = \{ (j, j-a) \}$

$a = 1, 2, 3, 4$

**Game 2:** $X = \{0, 1, 2, \cdots n\}^2$

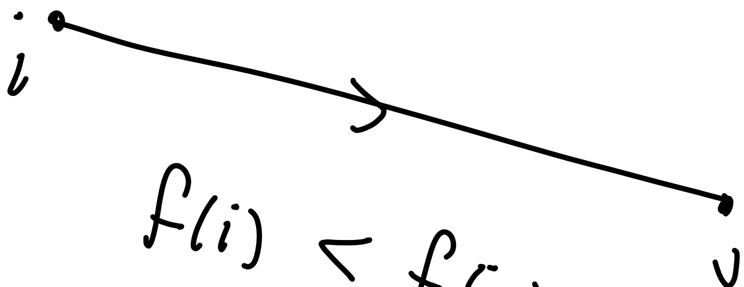
$A = \Big( (i, j), (i', j') \Big)$

$i = i'$ and $j' < j$

or $i' < i$ and $j' = j$

# Topological numbering:

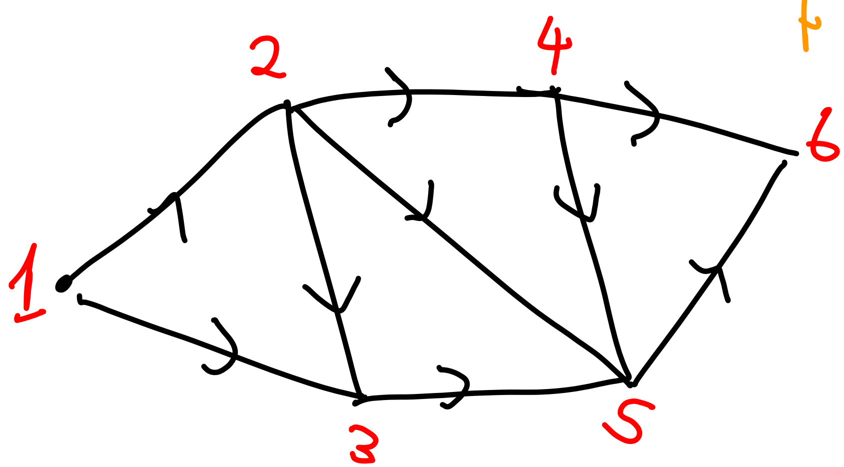
$f : X \longrightarrow [n]$ , $n = |X|$




$f(i) < f(j)$

If there is a topologically numbering then game must terminate

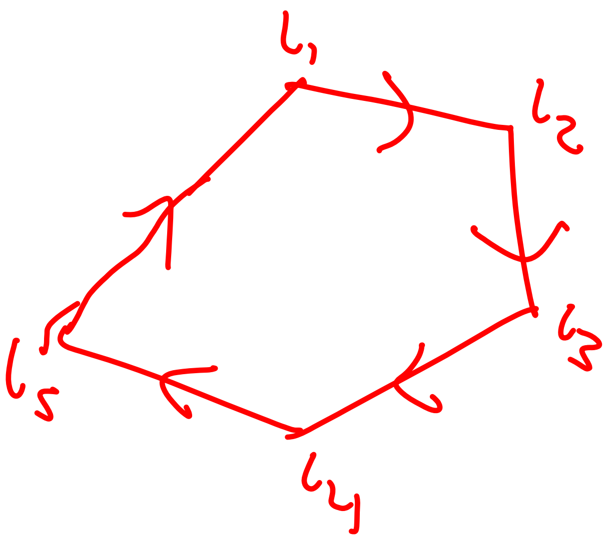
f values increase with each move,




f

A finite digraph has a topological numbering iff it has no directed cycles.

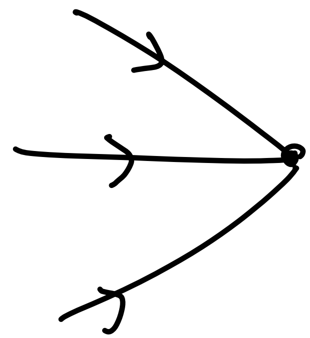
(1) Assume $f$ exists



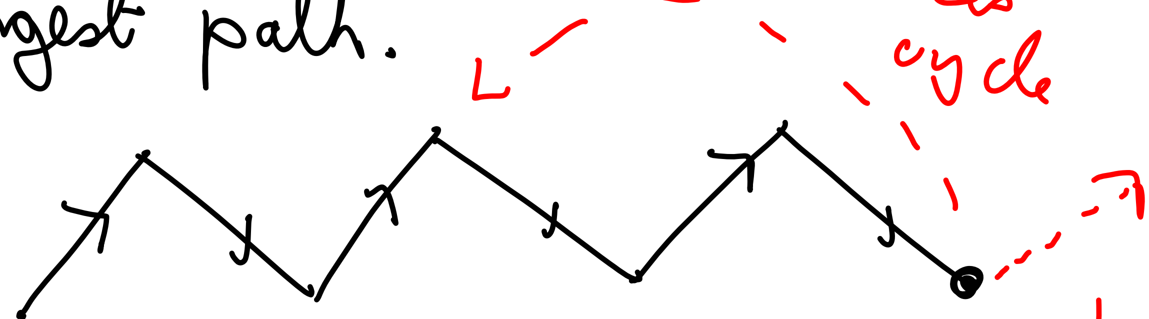

$$
\begin{aligned}
f(i_1) &< f(i_2) \\
&< f(i_3) \\
&< f(i_4) \\
&< f(i_5) \\
&< f(i_1) \quad ??
\end{aligned}
$$

(ii) Suppose $D$ is <u>acyclic</u>

$\exists$ a sink:




Take a longest path.



Now we use induction on $|X|$